Eigenvalue problem and a new product in cohomology of flag varieties

Prakash Belkale and Shrawan Kumar
Department of Mathematics
University of North Carolina
Chapel Hill, NC 27599–3250

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1 Introduction

Let G be a connected semisimple complex algebraic group and let P be a parabolic subgroup. In this paper we define a new (commutative and associative) product $\odot_0$ on the cohomology of the homogenous spaces G/P and use this to give a more efficient solution of the eigenvalue problem and also for the problem of determining the existence of G-invariants in the tensor product of irreducible representations of G. On the other hand, we show that this new product is intimately connected with the Lie algebra cohomology of the nil-radical of P via some works of Kostant and Kumar. We also initiate a uniform study of the geometric Horn problem for an arbitrary group G by obtaining two (a priori) different sets of necessary recursive conditions to determine when a cohomology product of Schubert classes in G/P is non-zero. Hitherto, this was studied largely only for the group $\mathrm{SL}(n)$ and its maximal parabolics P.

This new cohomology product is a certain deformation of the classical product. If $w \in W^P$ (W^P is the set of minimal length representatives in the cosets of W/W_P), let $[\bar{\Lambda}_w^P] \in H^*(G/P)$ be the cohomology class of the subvariety $\bar{\Lambda}_w^P := \overline{w^{-1}BwP} \subseteq G/P$. If the structure coefficients for the

classical product are written as

$$[\bar{\Lambda}_u^P] \cdot [\bar{\Lambda}_v^P] = \sum_w c_{u,v}^w [\bar{\Lambda}_w^P],$$

then the new product $\odot_0$ is a restricted sum

$$[\bar{\Lambda}_u^P] \odot_0 [\bar{\Lambda}_v^P] = {\sum_w}' c_{u,v}^w [\bar{\Lambda}_w^P],$$

where the summation is only over a smaller set of w which satisfy a certain numerical condition involving u, v and w (Definition 18). This numerical condition is best understood in terms of a notion which we call L-movability (c.f., Definition 4 and Section 5). If P is a minuscule maximal parabolic (e.g., for any maximal parabolic in $\mathrm{SL}(n)$), $\odot_0$ coincides with the classical product. However, even for any nonmaximal parabolic P in $\mathrm{SL}(n)$, $\odot_0$ differs from the classical product.

1.1 Eigenvalue problem

Choose a Borel subgroup B and a maximal torus $H \subset B$ of G. Let K be a maximal compact subgroup of G chosen such that $i\mathfrak{h}_{\mathbb{R}}$ is the Lie algebra of a maximal torus of K, where $\mathfrak{h}_{\mathbb{R}}$ is a real form of the Lie algebra $\mathfrak{h}$ of H. There is a natural homeomorphism $C : \mathfrak{k}/K \to \mathfrak{h}_+$, where K acts on $\mathfrak{k}$ by the adjoint representation and $\mathfrak{h}_+$ is the positive Weyl chamber in $\mathfrak{h}_{\mathbb{R}}$.

The *eigenvalue problem* is concerned with the following question:

(E) Determine all the s-tuples $(h_1, \dots, h_s) \in \mathfrak{h}_+^s$ for which there exist $(k_1, \dots, k_s) \in \mathfrak{k}^s$ such that $C(k_j) = h_j$ for $j = 1, \dots, s$, and

$$\sum_{j=1}^{s} k_j = 0.$$

Given a (standard) maximal parabolic subgroup P, let ω_P denote the corresponding fundamental weight. This is invariant under the Weyl group of P.

For (E), Leeb-Millson (following the works of Klyachko [Kl], Belkale [Bel1] and Berenstein-Sjamaar [BeSj]) obtained the following:

Let $(h_1, \cdots, h_s) \in \mathfrak{h}_+^s$. Then, the following are equivalent:

(A) $\exists(k_1, \cdots, k_s) \in \mathfrak{k}^s$ *such that* $k_1 + \cdots + k_s = 0$, *and* $C(k_j) = h_j$, *for* $j = 1, \cdots, s$.

(B) *For every standard maximal parabolic subgroup* P *in* G *and every choice of* s*-tuples* $(w_1, \cdots, w_s) \in (W^P)^s$ *such that*

$$[\bar{\Lambda}^P_{w_1}] \cdot \cdots \cdot [\bar{\Lambda}^P_{w_s}] = [\bar{\Lambda}^P_e] \in H^*(G/P, \mathbb{Z}),$$

the following inequality holds:

$$\omega_P(\sum_{j=1}^{s} w_j^{-1} h_j) \leq 0. \tag{1}$$

In our Theorem 28, we show that we can replace the condition (B) by a smaller set of inequalities. This is one of the main theorems of this paper.
Theorem: (A) *is equivalent to*

(B′) *For every standard maximal parabolic subgroup* P *in* G *and every choice of* s*-tuples* $(w_1, \cdots, w_s) \in (W^P)^s$ *such that*

$$[\bar{\Lambda}^P_{w_1}] \odot_0 \cdots \odot_0 [\bar{\Lambda}^P_{w_s}] = [\bar{\Lambda}^P_e] \in (H^*(G/P, \mathbb{Z}), \odot_0),$$

the inequality (1) holds.

Knutson-Tao-Woodward [KTW] proved that for $G = \mathrm{SL}(n)$ the inequalities (B) (which are, in this case, the same set of inequalities as (B′) because of Lemma 19) are irredundant. However, in the case of the groups of type B_3 and C_3, Kumar-Leeb-Millson [KLM] proved that of the total of 135 inequalities corresponding to the system (B), 33 are redundant. In contrast, making use of the tables in Section 10, it can be seen that none of the inequalities given by (B′) are redundant for either B_3 or C_3.

Let $\nu_1, \cdots, \nu_s$ be dominant integral weights. Then, consider the problem of finding conditions on ν_j's such that the space of G-invariants

$$H^0((G/B)^s, \mathcal{L}(N\nu_1) \boxtimes \cdots \boxtimes \mathcal{L}(N\nu_s))^G$$

is nonzero for some $N > 0$. As is well known, by using symplectic geometry, this problem is equivalent to the eigenvalue problem. For a precise solution of this problem, see Theorem 21.

1.2 Relation to Lie algebra cohomology

Using a celebrated theorem of Kostant on the cohomology of some nilpotent Lie algebras (Theorem 41), we relate the new cohomology product $\odot_0$ to Lie algebra cohomology. More specifically, in Theorem 42, we exhibit an explicit isomorphism of graded rings

$$\phi : \left(H^*(G/P, \mathbb{C}), \odot_0\right) \simeq \left[H^*(\mathfrak{u}_P) \otimes H^*(\mathfrak{u}_P^-)\right]^{\mathfrak{l}},$$

where we take the tensor product algebra structure on the right side. This isomorphism is used to determine the structure of $(H^*(G/B), \odot_0)$ completely (cf., Corollary 43).

1.3 Geometric Horn problem

One of the consequences of the work of Klyachko [Kl], and the saturation theorem of Knutson-Tao [KT] is that one can tell when a product of Schubert classes in a given Grassmannian $\mathrm{Gr}(r, n)$ (which is a homogenous space for $\mathrm{SL}(n)$ corresponding to a maximal parabolic) is nonzero by writing down a series of inequalities coming from knowing the answer to the same question for smaller Grassmannians. These inequalities are the eigenvalue inequalities (1) for $\mathrm{SL}(r)$ where the conjugacy classes are determined from the Schubert classes. We refer the reader to the survey article of Fulton [Fu2] for details.

In this paper we initiate a uniform study of solving the same problem for arbitrary semisimple groups.

We obtain two (a priori) different sets of necessary conditions (Theorems 29 and 36) for a product (with any number of factors) of Schubert classes to be nonzero in the cohomology of G/P. The inequalities given by Theorem 29, which we call *character inequalities*, are similar to the eigenvalue inequalities for the Levi subgroup L of P. These inequalities provide the first known numerical criterion for the vanishing of intersection multiplicities beyond the codimension condition for an arbitrary semisimple G.

The second set of inequalities, which we call the *dimension inequalities*, given by Theorem 36 are based on dimension counts. In the case of $G = \mathrm{SL}(n)$ and P a maximal parabolic, the dimension and character inequalities coincide. In general, even for minuscule maximal parabolics, these inequalities are (a priori) different.

The character inequalities can be refined further if we consider the new product $\odot_0$ instead of the classical product (Theorem 32). The character

inequalities for $\odot_0$ are sufficient for the homogenous spaces of the form G/B by virtue of Corollary 43.

1.4 Examples

In Section 10 we give the multiplication tables under the deformed product $\odot$ for G/P for all the rank 3 complex simple groups G and maximal parabolic subgroups P.

Sections 7,8 and 9 are independent of each other.

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2 Notation and Preliminaries on the Intersection Theory in G/P

Let G be a connected reductive complex algebraic group. We choose a Borel subgroup B and a maximal torus $H \subset B$ and let $W = W_G := N_G(H)/H$ be the associated Weyl group, where $N_G(H)$ is the normalizer of H in G. Let $P \supseteq B$ be a standard parabolic subgroup of G and let $U = U_P$ be its unipotent radical. Consider the Levi subgroup $L = L_P$ of P containing H, so that P is the semi-direct product of U and L. Then, $B_L := B \cap L$ is a Borel subgroup of L. Let $X(H)$ denote the character group of H, i.e., the group of all the algebraic group morphisms $H \to \mathbb{G}_m$. Then, B_L being the semidirect product of its commutator $[B_L, B_L]$ and H, any $\lambda \in X(H)$ extends uniquely to a character of B_L. Similarly, for any algebraic subgroup S of G, let $O(S)$ denote the set of all the one parameter subgroups in S, i.e., algebraic group morphisms $\mathbb{G}_m \to S$. We denote the Lie algebras of G, B, H, P, U, L, B_L by the corresponding Gothic characters: $\mathfrak{g}, \mathfrak{b}, \mathfrak{h}, \mathfrak{p}, \mathfrak{u}, \mathfrak{l}, \mathfrak{b}_L$ respectively. Let $R = R_{\mathfrak{g}}$ be the set of roots of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$ and let R^+ be the set of positive roots (i.e., the set of roots of $\mathfrak{b}$). Similarly, let $R_{\mathfrak{l}}$ be the set of roots of $\mathfrak{l}$ with respect to $\mathfrak{h}$ and $R^+_{\mathfrak{l}}$ be the set of roots of $\mathfrak{b}_L$. Let $\Delta = \{\alpha_1, \cdots, \alpha_\ell\} \subset R^+$ be the set of simple roots, where ℓ is the semisimple rank of G (i.e., the dimension of $\mathfrak{h}' := \mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]$). We denote by

$\Delta(P)$ the set of simple roots contained in $R_{\mathfrak{l}}$. For any $1 \leq j \leq \ell$, define the element $x_j \in \mathfrak{h}'$ by

$$\alpha_i(x_j) = \delta_{i,j}, \ \forall\ 1 \leq i \leq \ell. \tag{2}$$

For an H-invariant subspace $V \subset \mathfrak{g}$ under the adjoint action, let $R(V)$ denote the set of roots of $\mathfrak{g}$ appearing in V, i.e., $V = \oplus_{\alpha \in R(V)} \mathfrak{g}_\alpha$.

Recall that if W_P is the Weyl group of P (which is, by definition, the Weyl Group of L), then in each coset of W/W_P we have a unique member w of minimal length. This satisfies (cf. [Ku2, Exercise 1.3.E]):

$$wB_Lw^{-1} \subseteq B. \tag{3}$$

Let W^P be the set of the minimal length representatives in the cosets of W/W_P.

For any $w \in W^P$, define the (shifted) Schubert cell:

$$\Lambda_w^P := w^{-1}BwP \subset G/P.$$

Then, it is a locally closed subvariety of G/P isomorphic with the affine space $\mathbb{A}^{\ell(w)}, \ell(w)$ being the length of w (cf. [J, Part II, §13.1]). Its closure is denoted by $\bar{\Lambda}_w^P$, which is an irreducible (projective) subvariety of G/P of dimension $\ell(w)$. Considered as an element in the group of rational equivalence classes $A_{\ell(w)}(G/P)$ of algebraic degree $\ell(w)$ on G/P, it is denoted by $[\bar{\Lambda}_w^P]$. Since G/P is smooth, setting $A^*(G/P) := A_{\dim G/P - *}(G/P)$, $A^*(G/P)$ is a commutative associative graded ring under the intersection product $\cdot$ (cf. [Fu1, §8.3]).

Let $\mu(\bar{\Lambda}_w^P)$ denote the fundamental class of $\bar{\Lambda}_w^P$ considered as an element of the singular homology with integral coefficients $H_{\ell(w)}(G/P, \mathbb{Z})$ of G/P. Then, from the Bruhat decomposition, the elements $\{\mu(\bar{\Lambda}_w^P)\}_{w \in W^P}$ form a $\mathbb{Z}$-basis of $H_*(G/P, \mathbb{Z})$. Let $\{\epsilon_w^P\}_{w \in W^P}$ be the dual basis of the singular cohomology with integral coefficients $H^*(G/P, \mathbb{Z})$, i.e., for any $v, w \in W^P$ we have

$$\epsilon_v^P(\mu(\bar{\Lambda}_w^P)) = \delta_{v,w}.$$

By [Fu1, Example 19.1.11(b)] and [KLM, Proposition 2.6], there is a graded ring isomorphism, the cycle class map,

$$c : A^*(G/P) \to H^*(G/P, \mathbb{Z}), \ \ [\bar{\Lambda}_w^P] \mapsto \epsilon_{w_o w w_o^P}^P,$$

where w_o (resp. w_o^P) is the longest element of W (resp. W_P). (For $w \in W^P$, we have $w_o w w_o^P \in W^P$ by [KLM, Proposition 2.6].) *From now on, we will*

identify $A^*(G/P)$ with $H^*(G/P,\mathbb{Z})$ under c. Thus, under the identification c, $H^*(G/P,\mathbb{Z})$ has two $\mathbb{Z}$-bases: $\{[\bar{\Lambda}^P_w] = \epsilon^P_{w_o w w^P_o}\}_{w\in W^P}$ and $\{\epsilon^P_w\}_{w\in W^P}$.

Let $T^P = T(G/P)_e$ be the tangent space of G/P at $e \in G/P$. It carries a canonical action of P. For $w \in W^P$, define T^P_w to be the tangent space of Λ^P_w at e. We shall abbreviate T^P and T^P_w by T and T_w respectively when the reference to P is clear. By (3), B_L stabilizes Λ^P_w keeping e fixed. Thus,

$$B_L T_w \subset T_w. \tag{4}$$

Lemma 1. *$g\Lambda^P_w$ passes through $e \Leftrightarrow g\Lambda^P_w = p\Lambda^P_w$ for some $p \in P$.*

Proof. Since $g\Lambda^P_w$ passes through e, $g^{-1} \in \Lambda^P_w$, i.e., $g \in Pw^{-1}Bw$. Write $g = pw^{-1}bw$, for some $p \in P$ and $b \in B$. Then, $g\Lambda^P_w = pw^{-1}bw\Lambda^P_w = p\Lambda^P_w$. ☐

The following result is the starting point of our analysis.

Proposition 2. *Take any $s \geq 1$ and any $(w_1, \cdots, w_s) \in (W^P)^s$ such that*

$$\sum_{j=1}^{s} \operatorname{codim} \Lambda^P_{w_j} \leq \dim G/P. \tag{5}$$

(Clearly, (3) is equivalent to the following equation:

$$\sum_{j=1}^{s} \ell(w_j) \geq (s-1)\dim G/P.) \tag{6}$$

Then the following three conditions are equivalent:

(a) $[\bar{\Lambda}^P_{w_1}] \cdot \cdots \cdot [\bar{\Lambda}^P_{w_s}] \neq 0 \in A^*(G/P)$.

(Observe that, by the above isomorphism c, this is equivalent to the condition: $\epsilon^P_{w_o w_1 w^P_o} \cdots \epsilon^P_{w_o w_s w^P_o} \neq 0$.)

(b) *For generic $(p_1, \cdots, p_s) \in P^s$, the intersection*

$$p_1\Lambda^P_{w_1} \cap \cdots \cap p_s\Lambda^P_{w_s}$$

is transverse at e.

(c) *For generic $(p_1, \ldots, p_s) \in P^s$,*

$$\dim(p_1 T_{w_1} \cap \cdots \cap p_s T_{w_s}) = \dim G/P - \sum_{j=1}^{s} \operatorname{codim} \Lambda^P_{w_j}.$$

As proved below, the set of s-tuples in (b) as well as (c) is an open subset of P^s.

Proof. For $p \in P$ and $w \in W^P$, the tangent space to $p\Lambda_w^P$ at $e \in G/P$ is pT_w. Therefore, by the definition of transversality (cf. [S, Chap. II, §2.1]), (c) is equivalent to (b). The subset in (c) is the set of points $(p_1, \dots, p_s) \in P^s$ for which the canonical morphism

$$T \to \bigoplus_{j=1}^{s} \frac{T}{p_j T_{w_j}}$$

is surjective. Therefore, this set is open in P^s.

If $(p_1, \dots, p_s) \in P^s$ satisfies the property in (b), the smooth varieties $p_1\Lambda_{w_1}^P, \dots, p_s\Lambda_{w_s}^P$ meet transversally and hence properly at $e \in G/P$. By [Fu1, Proposition 7.1 and Section 12.2] this implies that (a) holds.

To show that (a) implies (b), find $g_j \in G$ for $j = 1, \dots, s$ so that $g_1\Lambda_{w_1}^P, \dots, g_s\Lambda_{w_s}^P$ meet transversally at a nonempty set of points (cf. Proposition 3). By translation, assume that $e \in G/P$ is one of these points. By Lemma 1, for any $j = 1, \cdots, s$, $g_j\Lambda_{w_j}^P = p_j\Lambda_{w_j}^P$, for some $p_j \in P$. Thus, we have found a point $(p_1, \dots, p_s) \in P^s$ satisfying the condition in (b). Since the condition is an open condition, we are assured of a nonempty open subset of P^s satisfying the property in (b). $\square$

Proposition 3 (Kleiman). *Let a connected algebraic group G act transitively on a smooth variety X and let $X_1, \dots, X_s$ be irreducible subvarieties of X. Then, there exists a non empty open subset $U \subseteq G^s$ such that for $(g_1, \dots, g_s) \in U$, the intersection $\bigcap_{j=1}^s g_jX_j$ is proper (possibly empty) and dense in $\bigcap_{j=1}^s g_j\bar{X}_j$.*

Moreover, if X_j, $j = 1, \dots, s$, are smooth varieties, we can find such a U with the additional property that for $(g_1, \dots, g_s) \in U$, $\bigcap_{j=1}^s g_jX_j$ is transverse at each point of intersection.

Proof. We include a proof of the density statement (the rest of the conclusion is standard, see [Kle]).

Let $Y_j := \bar{X}_j \setminus X_j$. Let U be a nonempty open subset of G^s such that for $(g_1, \dots, g_s) \in U$, the following intersections are proper:

1. $\cap_{j=1}^s g_j\bar{X}_j$.
2. For $\ell \in \{1, \dots, s\}$, $\{\cap_{j\in\{1,\dots,s\}\setminus\{\ell\}} g_j\bar{X}_j\} \cap g_\ell Y_\ell$.

For $(g_1, \ldots, g_s) \in U$ and $\ell \in \{1, \ldots, s\}$, each irreducible component of the intersection $\{\bigcap_{j\in\{1,\ldots,s\}\setminus\{\ell\}} g_j\bar{X}_j\} \cap g_\ell Y_\ell$ is therefore of dimension strictly less than that of each irreducible component of $\bigcap_{j=1}^s g_j\bar{X}_j$ (since $\dim(Y_\ell) < \dim(X_\ell)$). This proves the density statement. $\square$

3 Preliminary Analysis of Levi-movability

We begin by introducing the central concept of this paper:

Definition 4. Let $w_1, \cdots, w_s \in W^P$ be such that

$$\sum_{j=1}^{s} \operatorname{codim} \Lambda_{w_j}^P = \dim G/P. \tag{7}$$

This is equivalent to the condition:

$$\sum_{j=1}^{s} \ell(w_j) = (s-1)\dim G/P. \tag{8}$$

We then call the s-tuple $(w_1, \cdots, w_s)$ *Levi-movable* for short *L-movable* if, for generic $(l_1, \cdots, l_s) \in L^s$, the intersection $l_1\Lambda_{w_1} \cap \cdots \cap l_s\Lambda_{w_s}$ is transverse at e.

Observe that, even though in the definition of L-movability we took minimal coset representatives in W/W_P, our definition is independent of the choice of coset representatives. Thus, we have the notion of L-movability for any s-tuple $(w_1, \cdots, w_s) \in (W/W_P)^s$.

By Proposition 2, if $(w_1, \cdots, w_s)$ is L-movable, then $[\bar{\Lambda}_{w_1}^P] \cdot \cdots \cdot [\bar{\Lambda}_{w_s}^P] = d[\bar{\Lambda}_e^P]$ in $H^*(G/P)$, for some nonzero d. The converse is not true in general (cf., Theorem 15).

Definition 5. Let $w \in W^P$. Since $T_w := T_e(\Lambda_w)$ is a B_L-module (by (4)), we have the P-equivariant vector bundle $\mathcal{T}_w := P \underset{B_L}{\times} T_w$ on P/B_L associated to the principal B_L-bundle $P \to P/B_L$ via the B_L-module T_w. In particular, we have the P-equivariant vector bundle $\mathcal{T} := P \underset{B_L}{\times} T$ (where, as in Section 2, $T := T_e(G/P)$) and $\mathcal{T}_w$ is canonically a P-equivariant subbundle of $\mathcal{T}$. Take the top exterior powers $\det(\mathcal{T}/\mathcal{T}_w)$ and $\det(\mathcal{T}_w)$, which

are P-equivariant line bundles on P/B_L. Observe that, since T is a P-module, the P-equivariant vector bundle $\mathcal{T}$ is P-equivariantly isomorphic with the product bundle $P/B_L \times T$ under the map $\xi : P/B_L \times T \to \mathcal{T}$ taking $(pB_L, v) \mapsto (p, p^{-1}v)$ mod B_L, for $p \in P$ and $v \in T$; where P acts on $P/B_L \times T$ diagonally. We will often identify $\mathcal{T}$ with the product bundle $P/B_L \times T$ under ξ.

Similarly, for any $\lambda \in X(H)$, we have a P-equivariant line bundle $\mathcal{L}(\lambda) = \mathcal{L}_P(\lambda)$ on P/B_L associated to the principal B_L-bundle $P \to P/B_L$ via the one dimensional B_L-module λ^{-1}. (As observed in Section 2, any $\lambda \in X(H)$ extends uniquely to a character of B_L.) The twist in the definition of $\mathcal{L}(\lambda)$ is introduced so that the dominant characters correspond to the dominant line bundles.

For $w \in W^P$, define the character $\chi_w \in \mathfrak{h}^*$ by

$$\chi_w = \sum_{\beta \in (R^+ \setminus R^+_{\mathfrak{l}}) \cap w^{-1}R^+} \beta \,.$$

Then, from [Ku2, 1.3.22.3] and (3),

$$\chi_w = \rho - 2\rho^L + w^{-1}\rho, \tag{9}$$

where ρ (resp. ρ^L) is half the sum of roots in R^+ (resp. in $R^+_{\mathfrak{l}}$).

The following lemma is easy to establish.

Lemma 6. *For $w \in W^P$, as P-equivariant line bundles on P/B_L, we have:*

$$\det(\mathcal{T}/\mathcal{T}_w) = \mathcal{L}(\chi_w).$$

Observe that, since χ_1 is a P-module (as T is a P-module), the line bundle $\mathcal{L}(\chi_1)$ is a trivial line bundle. However, as an H-equivariant line bundle, it is nontrivial in general as the character χ_1 restricted to the connected center of L is nontrivial.

Let $\mathcal{T}_s$ be the P-equivariant product bundle $(P/B_L)^s \times T \to (P/B_L)^s$ under the diagonal action of P on $(P/B_L)^s \times T$. Then, $\mathcal{T}_s$ is canonically P-equivariantly isomorphic with the pull-back bundle $\pi_j^*(\mathcal{T})$, for any $1 \leq j \leq s$, where $\pi_j : (P/B_L)^s \to P/B_L$ is the projection onto the j-th factor. For any $w_1, \cdots, w_s \in W^P$, we have a P-equivariant map of vector bundles on $(P/B_L)^s$:

$$\Theta = \Theta_{(w_1, \cdots, w_s)} : \mathcal{T}_s \to \oplus_{j=1}^s \pi_j^*(\mathcal{T}/\mathcal{T}_{w_j}) \tag{10}$$

obtained as the direct sum of the canonical projections $\mathcal{T}_s \to \pi_j^*(\mathcal{T}/\mathcal{T}_{w_j})$ under the identification $\mathcal{T}_s \simeq \pi_j^*(\mathcal{T})$. Now, assume that $w_1, \cdots, w_s \in W^P$ satisfies the condition (7). In this case, we have the same rank on the two sides of the map (10). Let θ be the bundle map obtained from Θ by taking the top exterior power:

$$\theta = \det(\Theta) : \det \mathcal{T}_s \to \det\Big(\mathcal{T}/\mathcal{T}_{w_1}\Big) \boxtimes \cdots \boxtimes \det\big(\mathcal{T}/\mathcal{T}_{w_s}\big), \tag{11}$$

where $\boxtimes$ denotes the external tensor product. Clearly, θ is P-equivariant and hence one can view θ as a P-invariant element in

$$H^0\Bigg((P/B_L)^s, \det(\mathcal{T}_s)^* \otimes \Big(\det\big(\mathcal{T}/\mathcal{T}_{w_1}\big) \boxtimes \cdots \boxtimes \det\big(\mathcal{T}/\mathcal{T}_{w_s}\big)\Big)\Bigg).$$

$$= H^0\left((P/B_L)^s, \mathcal{L}(\chi_{w_1} - \chi_1) \boxtimes \cdots \boxtimes \mathcal{L}(\chi_{w_s})\right), \tag{12}$$

where the above equality follows from Lemma 6.

The following lemma is immediate:

Lemma 7. *Let $\bar{p} = (\bar{p}_1, \cdots, \bar{p}_s) \in (P/B_L)^s$. Then, under the assumption (7), the following are equivalent:*

1. *The restriction of the map Θ to the fiber over $\bar{p}$ is an isomorphism.*
2. *The section θ does not vanish at $\bar{p}$.*
3. *The locally-closed subvarieties $p_1\Lambda^P_{w_1}, \cdots, p_s\Lambda^P_{w_s}$ meet transversally at $e \in G/P$.*

The following corollary follows immediately from Proposition 2 and Lemma 7.

Corollary 8. *Let $(w_1, \cdots, w_s)$ be an s-tuple of elements of W^P satisfying the condition (7). Then, we have the following:*

1. *The section θ is nonzero if and only if*

$$[\bar{\Lambda}^P_{w_1}] \cdot \cdots \cdot [\bar{\Lambda}^P_{w_s}] \neq 0 \in H^*(G/P).$$

2. *The s-tuple $(w_1, \cdots, w_s)$ is L-movable if and only if the section θ restricted to $(L/B_L)^s$ is not identically* 0.

4 Geometric Invariant Theory Revisited

We need to consider the Geometric Invariant Theory (GIT) in a nontraditional setting, where a *nonreductive* group acts on a *nonprojective* variety. To handle such a situation, we need to introduce the notion of P-admissible one parameter subgroups (cf. Definition 11). But first we recall the following definition due to Mumford.

Definition 9. Let S be any (not necessarily reductive) algebraic group acting on a (not necessarily projective) variety $\mathbb{X}$ and let $\mathbb{L}$ be an S-equivariant line bundle on $\mathbb{X}$. Take any $x \in \mathbb{X}$ and a one parameter subgroup (for short OPS) $\lambda \in O(S)$ such that the limit

$$\lim_{t \to 0} \lambda(t)x$$

exists in $\mathbb{X}$ (i.e., the morphism $\lambda_x : \mathbb{G}_m \to X$ given by $t \mapsto \lambda(t)x$ extends to a morphism $\widetilde{\lambda}_x : \mathbb{A}^1 \to X$). This condition is satisfied by every $x \in \mathbb{X}$, if $\mathbb{X}$ is projective. Then, following Mumford, define a number $\mu^{\mathbb{L}}(x, \lambda)$ as follows: Let $\mathbb{L}' := \widetilde{\lambda}_x^*(\mathbb{L})$ be the pull-back line bundle on $\mathbb{A}^1$. Let σ_1 be a nonzero vector in the fiber of $\mathbb{L}$ over x. Then, using the $\mathbb{G}_m$-action, σ_1 extends to a $\mathbb{G}_m$-invariant section σ of $\mathbb{L}'$ over $\mathbb{G}_m$. Since $\mathbb{L}'$ is a line bundle on $\mathbb{A}^1$ we can speak of the order of vanishing $\mu^{\mathbb{L}}(x, \lambda)$ of σ at 0 (this number is negative if σ has a pole at $0 \in \mathbb{A}^1$).

Let V be a finite dimensional representation of S and let $i : \mathbb{X} \hookrightarrow \mathbb{P}(V)$ be an S-equivariant embedding. Take $\mathbb{L} := i^*(\mathcal{O}(1))$. Let $\lambda \in O(S)$ and let $\{e_1, \ldots, e_n\}$ be a basis of V consisting of eigenvectors, i.e., $\lambda(t) \cdot e_l = t^{\lambda_l} e_l$, for $l = 1, \ldots, n$. For any $x \in \mathbb{X}$ such that $\lim_{t \to 0} \lambda(t)x$ exists in $\mathbb{X}$, write $i(x) = [\sum_{l=1}^n x_l e_l]$. Then, it is easy to see that, we have ([MFK, Proposition 2.2.3, page 51])

$$\mu^{\mathbb{L}}(x, \lambda) = \max_{l : x_l \neq 0} (-\lambda_l). \tag{13}$$

We record the following simple properties of $\mu^{\mathbb{L}}(x, \lambda)$:

Proposition 10. *For any $x \in \mathbb{X}$ and $\lambda \in O(S)$ such that $\lim_{t \to 0} \lambda(t)x$ exists in $\mathbb{X}$, we have the following (for any S-equivariant line bundles $\mathbb{L}, \mathbb{L}_1, \mathbb{L}_2$):*

(a) $\mu^{\mathbb{L}_1 \otimes \mathbb{L}_2}(x, \lambda) = \mu^{\mathbb{L}_1}(x, \lambda) + \mu^{\mathbb{L}_2}(x, \lambda)$.

(b) If there exists $\sigma \in H^0(\mathbb{X}, \mathbb{L})^S$ such that $\sigma(x) \neq 0$, then $\mu^{\mathbb{L}}(x, \lambda) \geq 0$.

(c) If $\mu^{\mathbb{L}}(x,\lambda)=0$, then any element of $H^0(\mathbb{X},\mathbb{L})^S$ which does not vanish at x does not vanish at $\lim_{t\to 0}\lambda(t)x$ as well.

(d) For any S-variety $\mathbb{X}'$ together with an S-equivariant morphism $f:\mathbb{X}'\to\mathbb{X}$ and any $x'\in\mathbb{X}'$ such that $\lim_{t\to 0}\lambda(t)x'$ exists in $\mathbb{X}'$, we have

$$\mu^{f^*\mathbb{L}}(x',\lambda)=\mu^{\mathbb{L}}(f(x'),\lambda).$$

Definition 11. We call an OPS $\lambda\in O(P)$, *P-admissible* if the limit $\lim_{t\to 0}\lambda(t)x$ exists in P/B_L for every $x\in P/B_L$. When the reference to P is clear from the context, we will abbreviate P-admissible by *admissible*.

From the conjugacy of L and B_L, it is easy to see that the notion of the admissibility of λ does not depend upon the choice of the Levi subgroup L or the Borel subgroup B_L of L.

For an OPS $\lambda\in O(G)$, set

$$\dot\lambda:=\frac{d\lambda(t)}{dt}|_{t=1}\in\mathfrak{g},$$

its tangent vector.

Also define the *associated parabolic subgroup* $P(\lambda)$ of G by

$$P(\lambda):=\{g\in G:\lim_{t\to 0}\lambda(t)g\lambda(t)^{-1}\,\text{exists in}\,G\}.$$

We also denote $P(\lambda)$ sometimes by $P(\dot\lambda)$. This should not create any confusion since $\dot\lambda$ uniquely determines λ.

The following lemma gives a characterization of admissible one parameter subgroups in P.

Lemma 12. *Let $\lambda\in O(P)$. Then, λ is admissible iff*

$$P(\lambda)\supset U.$$

Equivalently, λ is admissible iff there exists $p\in P$ such that $\lambda_o:=p\lambda p^{-1}$ lies in H and $\beta(\dot\lambda_o)\geq 0$ for all $\beta\in R^+\setminus R^+_{\mathfrak{l}}$.

Proof. Clearly, λ is admissible iff any conjugate $p\lambda p^{-1}$ is admissible, for $p\in P$. Moreover, $P(p\lambda p^{-1})=pP(\lambda)p^{-1}$. Thus, U being normal in P, the condition that $P(\lambda)\supset U$ is invariant under the conjugation of λ via p.

Further, since any OPS in P can be conjugated via an element $p \in P$ inside the maximal torus H, we can assume that $\operatorname{Im} \lambda \subset H$. Decompose

$$P/B_L \simeq U \times L/B_L \text{ as varieties.}$$

For any OPS λ in H (in particular in L), and any $u \in U, l \in L$,

$$\lim_{t \to 0} \lambda(t)(ulB_L) = \lim_{t \to 0} \big(\lambda(t)u\lambda(t)^{-1}(\lambda(t)lB_L)\big).$$

But, since L/B_L is projective, the limit on the left side exists iff $\lim_{t \to 0} \lambda(t)u\lambda(t)^{-1}$ exists in U, which is equivalent to the condition that $P(\lambda) \supset U$. This proves the first part of the lemma.

The second part follows readily from the first part.

□

Definition 13. Let $x = ulB_L \in P/B_L$, for $u \in U$ and $l \in L$. Let λ be an admissible OPS and let $P_L(\lambda) := P(\lambda) \cap L$. Then, $P_L(\lambda)$ is a parabolic subgroup of L. To see this, write $\lambda = u'\lambda_o u'^{-1}$, for $u' \in U$ and with λ_o an OPS in L. This gives $P(\lambda) = u'P(\lambda_o)u'^{-1}$. But since λ is admissible by assumption, and hence so is λ_o; and thus by Lemma 12, $P(\lambda_o) \supset U$. This gives $P(\lambda) = P(\lambda_o)$. But, as is well known, $P_L(\lambda_o)$ is a parabolic subgroup of L and hence so is $P_L(\lambda)$.

Write $P_L(\lambda) = l_1 Q l_1^{-1}$, for some $l_1 \in L$, where Q is a standard parabolic subgroup of L, i.e., $Q \supset B_L$. Let $l^{-1}l_1 \in B_L w Q$, where $w \in W_L/W_Q$. Then, clearly $w \in W_L/W_Q$ does not depend upon the choices of the representatives l (in lB_L) and l_1. We define the *relative position* $[x, \lambda]$ to be $w \in W_L/W_Q$. This satisfies:

$$[px, p\lambda p^{-1}] = [x, \lambda], \text{ for any } p \in P. \tag{14}$$

Observe that, if λ is an OPS lying in the center of L, then for any $x \in P/B_L$,

$$[x, \lambda] = 1. \tag{15}$$

For an OPS λ in P, we can choose $p \in P$ such that $\lambda_o := p\lambda p^{-1}$ is an OPS in H and, moreover, $\dot{\lambda}_o \in \mathfrak{h}$ is L-dominant (i.e., $\alpha_i(\dot{\lambda}_o) \geq 0$ for all $\alpha_i \in \Delta(P)$). Set,

$$X_\lambda^P = \dot{\lambda}_o. \tag{16}$$

Then, X_λ^P is well defined. Moreover, W_Q fixes X_λ^P. We shall abbreviate X_λ^P by X_λ when the reference to P is clear.

The following lemma is a generalization of the corresponding result in [BeSj, Section 4.2].

Lemma 14. *Let λ be an admissible OPS in P, $x \in P/B_L$ and $\chi \in X(H)$. Then, we have the following formula:*

$$\mu^{\mathcal{L}(\chi)}(x,\lambda) = -\chi([x,\lambda]X_\lambda).$$

Proof. Let $p \in P$ be such that $\lambda = p\lambda_o p^{-1}$ (where λ_o is an OPS in H such that $\dot{\lambda}_o = X_\lambda$). Let $\bar{x} \in P$ be a lift of x. We seek an OPS $b(t)$ in B_L so that $\lambda_o(t)p^{-1}\bar{x}b(t)$ has a limit in P as $t \to 0$. Let $p = u_1 l_1, \bar{x} = ul$, for $u, u_1 \in U$ and $l, l_1 \in L$, and let Q be the standard parabolic subgroup of L as in Definition 13. Choose $w \in W_L/W_Q, b_l \in [B_L, B_L]$ and $q \in Q$ so that $l_1 = lb_l wq$. Then, by the definition,

$$w = [x,\lambda]. \tag{17}$$

Thus, for any $b(t) \in B_L$,

$$\lambda_o(t)p^{-1}\bar{x}b(t) = \lambda_o(t)l_1^{-1}u_1^{-1}ul_1l_1^{-1}lb(t) = \lambda_o(t)\hat{u}\lambda_o(t)^{-1}\lambda_o(t)q^{-1}\lambda_o(t)^{-1}\lambda_o(t)w^{-1}b_l^{-1}b(t),$$

where $\hat{u} := l_1^{-1}u_1^{-1}ul_1 \in U$. In view of Lemma 12 and since $P_L(\lambda_o) = Q$, we find that the OPS

$$b(t) := b_l w \lambda_o(t)^{-1} w^{-1}$$

"works", i.e., $\lambda_o(t)p^{-1}\bar{x}b(t)$ has a limit in P as $t \to 0$.

Consider the $\mathbb{G}_m$-invariant section $\sigma(t) := (\lambda(t)\bar{x}, 1) \bmod B_L$ of $\lambda_x^*(\mathcal{L}_\chi)$ over $\mathbb{G}_m$, where, as in Definition 9, $\lambda_x : \mathbb{G}_m \to P/B_L$ is the map $t \mapsto \lambda(t)\cdot x$. Then, the section $\sigma(t)$ corresponds to the function $\mathbb{G}_m \to \mathbb{A}^1$, $t \mapsto \chi^{-1}(b(t)^{-1})$. From this we get the lemma by using (17). $\square$

5 Criterion for L-movability

The aim of this section is to prove the following characterization of L-movability.

Theorem 15. *Assume that $(w_1, \cdots, w_s) \in (W^P)^s$ satisfies equation (7). Then, the following are equivalent.*

(a) $(w_1, \cdots, w_s)$ is L-movable.

(b) $[\bar{\Lambda}_{w_1}^P] \cdot \cdots \cdot [\bar{\Lambda}_{w_s}^P] = d[\bar{\Lambda}_e^P]$ in $H^*(G/P)$, for some nonzero d, and for each $\alpha_i \in \Delta \setminus \Delta(P)$, we have

$$\Big((\sum_{j=1}^{s} \chi_{w_j}) - \chi_1\Big)(x_i) = 0,$$

where χ_w is as defined in Definition 5 and $x_i \in \mathfrak{h}$ is defined by (2).

Proof. (a)⇒(b): Let $(w_1, \cdots, w_s) \in (W^P)^s$ be L-movable. Consider the restriction $\hat{\theta}$ of the P-invariant section

$$\theta \in H^0\left((P/B_L)^s, \mathcal{L}(\chi_{w_1} - \chi_1) \boxtimes \cdots \boxtimes \mathcal{L}(\chi_{w_s})\right)$$

to $(L/B_L)^s$, where θ is as defined in equations (11)–(12). Then, $\hat{\theta}$ is non-vanishing by Corollary 8. But for

$$H^0\left((L/B_L)^s, \mathcal{L}(\chi_{w_1} - \chi_1) \boxtimes \cdots \boxtimes \mathcal{L}(\chi_{w_s})\right)^L$$

to be non-empty, the center of L should act trivially (under the diagonal action) on $\mathcal{L}(\chi_{w_1} - \chi_1) \boxtimes \cdots \boxtimes \mathcal{L}(\chi_{w_s})$ restricted to $(L/B_L)^s$. This gives

$$\sum_{j=1}^{s} \chi_{w_j}(h) = \chi_1(h),$$

for all h in the Lie algebra $\mathfrak{z}_L$ of the center of L. For any $\alpha_i \in \Delta \setminus \Delta(P)$, $h = x_i$ clearly lies in $\mathfrak{z}_L$. Taking $h = x_i$ in the above equation, we obtain the implication (a)⇒(b) by using Corollary 8.

We now prove the implication (b)⇒(a). Thus, we are given that for each $\alpha_i \in \Delta \setminus \Delta(P)$,

$$\big((\sum_{j=1}^{s} \chi_{w_j}) - \chi_1\big)(x_i) = 0.$$

Moreover, by Corollary 8, $\theta(\bar{p}_1, \cdots, \bar{p}_s) \neq 0$, for some $\bar{p}_j \in P/B_L$. Consider the central OPS of L: $\lambda(t) := \prod_{\alpha_i \in \Delta \setminus \Delta(P)} t^{x_i}$. Clearly,

$$\dot{\lambda} = \sum_{\alpha_i \in \Delta \setminus \Delta(P)} x_i. \tag{18}$$

Thus, by Lemma 12, λ is admissible. For any $x = ulB_L \in P/B_L$, with $u \in U$ and $l \in L$,

$$\lim_{t \to 0} \lambda(t)x = \lim_{t \to 0} \lambda(t)u\lambda(t)^{-1}(\lambda(t)l)B_L.$$

But, since $\beta(\dot{\lambda}) > 0$, for all $\beta \in R^+ \setminus R^+_{\mathfrak{l}}$, we get

$$\lim_{t \to 0} \lambda(t)u\lambda(t)^{-1} = 1.$$

Moreover, since $\lambda(t)$ is central in L, the limit $\lim_{t \to 0} \lambda(t)lB_L$ exists and equals lB_L. Thus, $\lim_{t \to 0} \lambda(t)x$ exists and lies in L/B_L.

Now, let $\mathbb{L}$ be the P-equivariant line bundle $\mathcal{L}(\chi_{w_1}-\chi_1)\boxtimes\cdots\boxtimes\mathcal{L}(\chi_{w_s})$ on $\mathbb{X} := (P/B_L)^s$, and $\bar{p} := (\bar{p}_1, \cdots, \bar{p}_s) \in \mathbb{X}$. Then, by Lemma 14 and equations (15) and (18), we get

$$\begin{aligned}\mu^{\mathbb{L}}(\bar{p},\lambda) &= (\chi_1 - \chi_{w_1})([\bar{p}_1,\lambda]\dot{\lambda}) - \sum_{j=2}^{s} \chi_{w_j}([\bar{p}_j,\lambda]\dot{\lambda}) \\ &= -\sum_{\alpha_i\in\Delta\setminus\Delta(P)} \Big(\big((\sum_{j=1}^{s}\chi_{w_j}) - \chi_1\big)(x_i)\Big) \\ &= 0, \text{ by assumption.}\end{aligned}$$

Therefore, using Proposition 10(c) for $S = P$, θ does not vanish at $\lim_{t\to 0}\lambda(t)\bar{p}$. But, from the above, this limit exists as an element of $(L/B_L)^s$. Hence, $(w_1, \cdots, w_s)$ is L-movable by Corollary 8.

□

6 Deformation of Cup Product in $H^*(G/P)$

Define the structure constants $c_{u,v}^w$ under the intersection product in $H^*(G/P,\mathbb{Z})$ via the formula

$$[\bar{\Lambda}_u^P]\cdot[\bar{\Lambda}_v^P] = \sum_{w\in W^P} c_{u,v}^w [\bar{\Lambda}_w^P]. \tag{19}$$

The number $c_{u,v}^w$ is the number of points (counted with multiplicity) in the intersection $g_1[\bar{\Lambda}_u^P]\cap g_2[\bar{\Lambda}_v^P]\cap g_3[\bar{\Lambda}^P_{w_o w w_o^P}]$ for generic $(g_1,g_2,g_3)\in G^3$. If the generic intersection is infinite, we set $c_{u,v}^w = 0$.

Introduce the indeterminates τ_i for each $\alpha_i \in \Delta\setminus\Delta(P)$ and write a deformed cup product

$$[\bar{\Lambda}_u^P]\odot[\bar{\Lambda}_v^P] = \sum_{w\in W^P}\Big(\prod_{\alpha_i\in\Delta\setminus\Delta(P)} \tau_i^{(\chi_w-(\chi_u+\chi_v))(x_i)}\Big) c_{u,v}^w [\bar{\Lambda}_w^P], \tag{20}$$

where χ_w is defined in Definition 5. Extend this to a $\mathbb{Z}[\tau_i]_{\alpha_i\in\Delta\setminus\Delta(P)}$–linear product structure on $H^*(G/P,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}[\tau_i]$. By the next Proposition 17, if $c_{u,v}^w \neq 0$,

$$(\chi_w - (\chi_u+\chi_v))(x_i) \geq 0, \quad \text{for any } \alpha_i \in \Delta\setminus\Delta(P)$$

and hence the above product indeed lies in $H^*(G/P,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}[\tau_i]$. Clearly, $\odot$ is commutative. *This product should not be confused with the product in the quantum cohomology of G/P.*

Recall the definition of T, T_w from Section 2.

Lemma 16. *For any $w \in W^P$,*
(a) $T_w \oplus w_o^P(T_{w_o w w_o^P}) = T$.
(b) $\chi_w + w_o^P \chi_{w_o w w_o^P} = \chi_1$.
(c) $(\chi_w + \chi_{w_o w w_o^P} - \chi_1)(x_i) = 0$, for all $\alpha_i \in \Delta \setminus \Delta(P)$.
(d) For any $v, w \in W^P$ such that $\ell(v) = \ell(w)$, $[\bar{\Lambda}_v^P] \odot [\bar{\Lambda}_{w_o w w_o^P}^P] = \delta_{v,w}[\bar{\Lambda}_1^P]$.

Proof. As in Section 2, let $R(T_w)$ denote the set of roots α such that the root space $\mathfrak{g}_\alpha \subset T_w$, i.e., $T_w = \oplus_{\alpha \in R(T_w)} \mathfrak{g}_\alpha$. Then,

$$R(T_w) = w^{-1}R^+ \cap \big(R^- \setminus R_{\mathfrak{l}}^-\big). \tag{21}$$

To prove (a), it suffices to show that we have the disjoint union:

$$R^- \setminus R_{\mathfrak{l}}^- = R(T_w) \sqcup w_o^P \cdot R(T_{w_o w w_o^P}). \tag{22}$$

Since w_o^P keeps $R^- \setminus R_{\mathfrak{l}}^-$ stable, by (21),

$$\begin{aligned} w_o^P \cdot R(T_{w_o w w_o^P}) &= w^{-1} w_o R^+ \cap \big(R^- \setminus R_{\mathfrak{l}}^-\big) \\ &= w^{-1} R^- \cap \big(R^- \setminus R_{\mathfrak{l}}^-\big). \end{aligned}$$

From this and equation (21), equation (22) follows.

To prove (b), by equation (9),

$$\begin{aligned} \chi_w + w_o^P \chi_{w_o w w_o^P} &= \rho - 2\rho^L + w^{-1}\rho + w_o^P \rho - 2 w_o^P \rho^L + w^{-1} w_o \rho \\ &= \rho - 2\rho^L + w^{-1}\rho + w_o^P(\rho - \rho^L) - w_o^P \rho^L - w^{-1}\rho \\ &= \rho - 2\rho^L + \rho - \rho^L + \rho^L, \text{ since } w_o^P \text{ permutes } R^+ \setminus R_{\mathfrak{l}}^+ \\ &= 2\rho - 2\rho^L \\ &= \chi_1. \end{aligned}$$

This proves (b).

For any $\alpha_i \in \Delta \setminus \Delta(P)$, x_i is central in $\mathfrak{l}$; in particular, w_o^P acts trivially on x_i. Thus (c) follows from (b).

By [KLM, Lemma 2.9],

$$[\bar{\Lambda}_v^P] \cdot [\bar{\Lambda}_{w_o w w_o^P}^P] = \delta_{v,w}[\bar{\Lambda}_1^P]. \tag{23}$$

Thus, (d) follows from the defining equation (20) and the (c)-part.

∎

Proposition 17. *(a) For any $u, v, w \in W^P$ such that $c_{u,v}^w \neq 0$, we have*

$$(\chi_w - \chi_u - \chi_v)(x_i) \geq 0, \quad \textit{for each } \alpha_i \in \Delta \setminus \Delta(P). \tag{24}$$

(b) The product $\odot$ in $H^*(G/P) \otimes \mathbb{Z}[\tau_i]$ is associative.

(c) For $(w_1, \dots, w_s) \in (W^P)^s$ and $w \in W^P$, the coefficient of $[\bar{\Lambda}_w^P]$ in

$$[\bar{\Lambda}_{w_1}^P] \odot \cdots \odot [\bar{\Lambda}_{w_s}^P]$$

is

$$\prod_{\alpha_i \in \Delta \setminus \Delta(P)} \tau_i^{(\chi_w - \sum_{j=1}^s \chi_{w_j})(x_i)}$$

times the coefficient of $[\bar{\Lambda}_w^P]$ in the usual cohomology product

$$[\bar{\Lambda}_{w_1}^P] \cdot \cdots \cdot [\bar{\Lambda}_{w_s}^P].$$

Proof. By equation (23), $c_{u,v}^w \neq 0$ iff

$$[\bar{\Lambda}_u^P] \cdot [\bar{\Lambda}_v^P] \cdot [\bar{\Lambda}_{w_o w w_o^P}^P] = d[\bar{\Lambda}_1^P],$$

for some nonzero d. Thus, by Theorem 29(a), for any $\alpha_i \in \Delta \setminus \Delta(P)$,

$$(\chi_u + \chi_v + \chi_{w_o w w_o^P} - \chi_1)(x_i) \leq 0.$$

By Lemma 16(c), this gives

$$(\chi_u + \chi_v - \chi_w)(x_i) \leq 0,$$

proving (a).

To prove (b), write

$$\begin{aligned}
\big([\bar{\Lambda}_u^P] \odot [\bar{\Lambda}_v^P]\big) \odot [\bar{\Lambda}_w^P] &= \sum_{\theta \in W^P} \big(\prod \tau_i^{(\chi_\theta - (\chi_u + \chi_v))(x_i)}\big) c_{u,v}^\theta [\bar{\Lambda}_\theta^P] \odot [\bar{\Lambda}_w^P] \\
&= \sum_{\theta, \eta} \big(\prod \tau_i^{(\chi_\eta - \chi_w - \chi_u - \chi_v)(x_i)}\big) c_{u,v}^\theta c_{\theta,w}^\eta [\bar{\Lambda}_\eta^P].
\end{aligned}$$

Similarly,

$$[\bar{\Lambda}_u^P] \odot \big([\bar{\Lambda}_v^P] \odot [\bar{\Lambda}_w^P]\big) = \sum_{\theta, \eta} \big(\prod \tau_i^{(\chi_\eta - \chi_u - \chi_v - \chi_w)(x_i)}\big) c_{u,\theta}^\eta c_{v,w}^\theta [\bar{\Lambda}_\eta^P].$$

So, the associativity of $\odot$ follows from the corresponding associativity in $H^*(G/P)$ under the standard cup product. The property (c) is immediate from the definitions and property (b). $\square$

Definition 18. The cohomology of G/P obtained by setting each $\tau_i = 0$ in $(H^*(G/P,\mathbb{Z}) \otimes \mathbb{Z}[\tau_i], \odot)$ is denoted by $(H^*(G/P,\mathbb{Z}), \odot_0)$. Thus, as a $\mathbb{Z}$-module, this is the same as the singular cohomology $H^*(G/P,\mathbb{Z})$. This has essentially the effect of ignoring all the non L-movable intersections. By the above proposition, $(H^*(G/P,\mathbb{Z}), \odot_0)$ is associative (and commutative). Moreover, by Lemma 16(d), it continues to satisfy the Poincaré duality.

Lemma 19. *Let P be a minuscule maximal standard parabolic subgroup of G (i.e., the simple root $\alpha_p \in \Delta \setminus \Delta(P)$ appears with coefficient 1 in the highest root of R^+ and hence in all the roots in $R(\mathfrak{u}_P)$). Then, for any $u, v \in W^P$,*

$$[\bar{\Lambda}_u^P] \odot [\bar{\Lambda}_v^P] = [\bar{\Lambda}_u^P] \cdot [\bar{\Lambda}_v^P].$$

Proof. By the definition of $\odot$, it suffices to show that for any $w \in W^P$ such that $c_{u,v}^w \neq 0$,

$$(\chi_w - (\chi_u + \chi_v))(x_p) = 0. \tag{25}$$

By the definition of χ_w (cf. Definition 5), since P is minuscule,

$$\chi_w(x_p) = \mid w^{-1}R^+ \cap \left(R^+ \setminus R_{\mathfrak{l}}^+\right) \mid = \operatorname{codim}(\Lambda_w^P; G/P), \tag{26}$$

where the last equality follows from equation (21). Moreover, since $c_{u,v}^w \neq 0$,

$$\operatorname{codim}(\Lambda_u^P; G/P) + \operatorname{codim}(\Lambda_v^P; G/P) = \operatorname{codim}(\Lambda_w^P; G/P). \tag{27}$$

Combining equations (26) and (27), we get equation (25). ☐

7 Solution of the Eigenvalue Problem

Our aim in this section is to prove that for the solution of the eigenvalue problem one may restrict to those inequalities coming from L-movable intersections with intersection number one.

7.1 Principal criterion for the nontriviality of the space of invariants

Let $\nu_1, \cdots, \nu_s \in X(H)$ be dominant weights and let $\mathbb{L}$ be the G-linearized line bundle $\mathcal{L}(\nu_1) \boxtimes \cdots \boxtimes \mathcal{L}(\nu_s)$ on $(G/B)^s$.

In this subsection, we give our principal criterion to decide whether there exists an integer $N > 0$ such that $H^0((G/B)^s, \mathbb{L}^N)^G \neq 0$.

Let $P_1, \cdots, P_s$ be the standard parabolic subgroups such that the line bundle $\mathcal{L}(\nu_j)$ on G/B descends as an ample line bundle on G/P_j, still denoted by $\mathcal{L}(\nu_j)$. Consider $\mathbb{X} := G/P_1 \times \cdots \times G/P_s$ with the diagonal action of G and $\mathbb{L}$ the G-linearized ample line bundle $\mathcal{L}(\nu_1) \boxtimes \cdots \boxtimes \mathcal{L}(\nu_s)$ on $\mathbb{X}$. Let $\pi : (G/B)^s \to \mathbb{X}$ be the canonical projection map.

Lemma 20. *Let $x \in (G/B)^s$ and let $x' = \pi(x)$. The following are equivalent.*

(a) For some integer $N > 0$, there exists $\sigma \in H^0((G/B)^s, \mathbb{L}^N)^G$ such that $\sigma(x) \neq 0$, where G acts diagonally.

(b) For every OPS $\lambda \in O(G)$, $\mu^{\mathbb{L}}(x, \lambda) \geq 0$.

(c) For every OPS $\lambda \in O(G)$, $\mu^{\mathbb{L}}(x', \lambda) \geq 0$.

(d) For some integer $N > 0$, there exists $\sigma' \in H^0(\mathbb{X}, \mathbb{L}^N)^G$ such that $\sigma'(x') \neq 0$, i.e., x' is a semistable point of $\mathbb{X}$ (with respect to the G-linearized ample line bundle $\mathbb{L}$).

Proof. We first note that for any integer $N > 0$, the map π induces an isomorphism of G-modules:

$$H^0(\mathbb{X}, \mathbb{L}^N) \simeq H^0((G/B)^s, \mathbb{L}^N). \tag{28}$$

It follows from Proposition 10(d) that (b) is equivalent to (c).

By Hilbert-Mumford theorem [MFK, Theorem 2.1], since $\mathbb{L}$ is ample on $\mathbb{X}$, (c) is equivalent to (d).

By equation (28), (d) is equivalent to (a) and we are done. $\square$

We now state one of the main theorems of this paper.

Theorem 21. *Let G be a connected semisimple group. With the notation as above, the following are equivalent:*

(i) For some integer $N > 0$,

$$H^0((G/B)^s, \mathbb{L}^N)^G \neq 0.$$

*(ii) For every standard maximal parabolic subgroup P in G and every choice of L-**movable** s-tuples $(w_1, \cdots, w_s) \in (W^P)^s$ such that*

$$[\bar{\Lambda}^P_{w_1}] \cdot \cdots \cdot [\bar{\Lambda}^P_{w_s}] = [\bar{\Lambda}^P_e] \in H^*(G/P, \mathbb{Z}),$$

the following inequality holds:

$$\sum_{j=1}^{s} \nu_j(w_j x_{i_P}) \leq 0, \tag{29}$$

where α_{i_P} is the simple root in $\Delta \setminus \Delta(P)$.

(iii) For every standard maximal parabolic subgroup P in G and every choice of s-tuples $(w_1, \cdots, w_s) \in (W^P)^s$ such that

$$[\bar{\Lambda}_{w_1}^P] \odot_0 \cdots \odot_0 [\bar{\Lambda}_{w_s}^P] = [\bar{\Lambda}_e^P] \in \big(H^*(G/P, \mathbb{Z}), \odot_0\big),$$

the above inequality (29) holds.

The proof of this theorem will be given in subsection 7.3.

Remark 22. (a) The above theorem remains true for any connected reductive G provided we assume that $\sum_j \nu_{j|\mathfrak{z}(\mathfrak{g})} = 0$, where $\mathfrak{z}(\mathfrak{g})$ is the center of $\mathfrak{g}$.

(b) The implication $(i) \Rightarrow (ii)$ (and hence also the implication $(i) \Rightarrow (iii)$) in the above theorem rermains true for any (not necessarily maximal) standard parabolic P of G.

7.2 Maximally destabilizing one parameter subgroups

We recall the definition of the Kempf's OPS attached to an unstable point, which is in some sense 'most destabilizing' OPS. The exposition below follows the paper of Hesselink [H].

Let X be a projective variety with the action of a reductive group G and let $\mathcal{L}$ be a G-linearized ample line bundle on X.

We introduce the set $M(G)$ of fractional OPS in G. This is the set consisting of the ordered pairs (δ, a), where $\delta : \mathbb{G}_m \to G$ is an OPS of G and $a \in \mathbb{Z}_{>0}$, modulo the equivalence relation $(\delta, a) \simeq (\nu, b)$ if $\delta^b = \nu^a$. An OPS δ of G gives the element $(\delta, 1) \in M(G)$. The Killing form induces a norm q on $M(G)$ satisfying $aq(\delta, a) = q(\delta, 1)$, and $q(\delta, 1) = \|\dot{\delta}\|$.

We can extend the definition of $\mu^{\mathcal{L}}(x, \lambda)$ for any element $\lambda = (\delta, a) \in M(G)$ and $x \in X$ by setting $\mu^{\mathcal{L}}(x, \lambda) = \frac{\mu^{\mathcal{L}}(x,\delta)}{a}$.

For any OPS λ of G, recall the definition of the associated parabolic subgroup $P(\lambda)$ of G from Definition 11. We extend the definition of $P(\lambda)$ for any $\lambda = (\delta, a) \in M(G)$ by setting

$$P(\lambda) = P(\delta).$$

Then, the group

$$L(\lambda) := \{p \in G \mid \lim_{t\to 0} \lambda(t)p\lambda(t)^{-1} = p\}$$

is a Levi subgroup of $P(\lambda)$ and, moreover,

$$U(\lambda) := \{p \in G \mid \lim_{t\to 0} \lambda(t)p\lambda(t)^{-1} = e \in G\}$$

is the unipotent radical of $P(\lambda)$.

We note the following elementary property: If $\lambda \in M(G)$ and $p \in P(\lambda)$ then

$$\mu^{\mathcal{L}}(x, \lambda) = \mu^{\mathcal{L}}(x, p\lambda p^{-1}). \tag{30}$$

For any unstable point $x \in X$, define

$$q^*(x) = \inf_{\lambda \in M(G)} \{q(\lambda) \mid \mu^{\mathcal{L}}(x, \lambda) \leq -1\},$$

and the *optimal class*

$$\Lambda(x) = \{\lambda \in M(G) \mid \mu^{\mathcal{L}}(x, \lambda) \leq -1, q(\lambda) = q^*(x)\}.$$

Any $\lambda \in \Lambda(x)$ is called *Kempf's OPS associated to x.*

The following theorem is due to Kempf (cf. [K, Lemma 12.13]).

Theorem 23. *For any unstable point $x \in X$ and λ_1, $\lambda_2 \in \Lambda(x)$, $P(\lambda_1) = P(\lambda_2)$. Moreover, there exist $p_1, p_2 \in P(\lambda_1)$ so that $p_1\lambda_1p_1^{-1} = p_2\lambda_2p_2^{-1}$.*

Conversely, for $\lambda \in \Lambda(x)$ and $p \in P(\lambda)$, we have $p\lambda p^{-1} \in \Lambda(x)$ by equation (30).

The parabolic $P(\lambda)$ for $\lambda \in \Lambda(x)$ will be denoted by $P(x)$ and called the *Kempf's parabolic associated to the unstable point x.* We recall the following theorem due to Ramanan-Ramanathan [RR].

Theorem 24. *For any unstable point $x \in X$ and $\lambda = (\delta, a) \in \Lambda(x)$, let*

$$x_o = \lim_{t\to 0} \delta(t) \cdot x \in X.$$

Then, x_o is unstable and $\lambda \in \Lambda(x_o)$.

Remark 25. The above results are valid for points $x \in (G/B)^s$ (for the linearization $\mathbb{L}$ as in subsection 7.1). We may see this by applying Theorems 23, 24 to the image $x' \in \mathbb{X}$ (and using Proposition 10(d)).

7.3 Proof of Theorem 21

We return to the notation and assumptions of subsection 7.1. In particular, $\nu_1, \cdots, \nu_s \in X(H)$ are dominant weights and $\mathbb{L}$ denotes the G-linearized line bundle $\mathcal{L}(\nu_1) \boxtimes \cdots \boxtimes \mathcal{L}(\nu_s)$ on $(G/B)^s$. Also, $P_1, \cdots, P_s$ are the standard parabolic subgroups such that $\mathbb{L}$ descends as an ample line bundle (still denoted by) $\mathbb{L}$ on $\mathbb{X} := G/P_1 \times \cdots \times G/P_s$. We call a point $x \in (G/B)^s$ *semistable* (with respect to, not necessarily ample, $\mathbb{L}$) if its image in $\mathbb{X}$ under the canonical map $\pi : (G/B)^s \to \mathbb{X}$ is semistable.

By Lemma 20 and equation (28), condition (i) of Theorem 21 is equivalent to the following condition:

(iv) The set of semistable points of $(G/B)^s$ with respect to $\mathbb{L}$ is non empty.

Moreover, by the definition of $\odot_0$ and Theorem 15, the conditions (ii) and (iii) of Theorem 21 are equivalent.

Proof of the implication (iv) $\Rightarrow$ (ii) **of Theorem 21:** Let $x = (\bar{g}_1, \ldots, \bar{g}_s) \in (G/B)^s$ be a semistable point, where $\bar{g}_j = g_j B$. Since the set of semistable points is clearly open, we can choose a generic enough x such that the intersection $\cap g_j B w_j P$ itself is nonempty. (By assumption, $\cap \overline{g_j B w_j P}$ is nonempty for any g_j.) Pick $f \in \cap g_j B w_j P$. Consider the OPS $\lambda = t^{x_{i_P}}$ which is central in L. Since $P(\lambda) \supset L$ and, clearly, $P(\lambda) \supset B$, we have $P(\lambda) \supset P$. But, by assumption, P is a maximal parabolic subgroup and hence $P(\lambda) = P$ (it is easy to see that $P(\lambda) \neq G$). Moreover, from Definition 13 applied to the case $P = G$, it is easy to see that $[\bar{g}_j, f\lambda f^{-1}] = w_j$ and, $X_{f\lambda f^{-1}} = x_{i_P}$. Thus, applying Lemma 14 for $P = G$, the required inequality (29) is the same as $\mu^{\mathbb{L}}(x, f\lambda f^{-1}) \geq 0$, but this follows from Lemma 20, since x is semistable by assumption.

Before we come to the proof of the implication (ii) $\Rightarrow$ (i) in Theorem 21, we need to recall the following result due to Leeb-Millson.

Suppose that $x = (\bar{g}_1, \ldots, \bar{g}_s) \in (G/B)^s$ is an unstable point and $P(x)$ the Kempf's parabolic associated to x. Let $\lambda = (\delta, a)$ be a Kempf's OPS associated to x. Express $\delta(t) = f\gamma(t)f^{-1}$, where $\dot{\gamma} \in \mathfrak{h}_+$. Then $P(\gamma)$ is a standard parabolic. Let P be a maximal parabolic containing $P(\gamma)$. Define $w_j \in W/W_{P(\gamma)}$ by $fP(\gamma) \in g_j B w_j P(\gamma)$ for $j = 1, \ldots, s$. We recall the following theorem due to Leeb-Millson applicable to any unstable point $x \in (G/B)^s$. We postpone the proof of this theorem to the next subsection.

Theorem 26. *(a) The intersection $\bigcap_{j=1}^s g_j B w_j P \subset G/P$ is the singleton $\{fP\}$.*

(b) For the simple root $\alpha_{i_P} \in \Delta \setminus \Delta(P)$, $\sum_{j=1}^s \nu_j(w_j x_i) > 0$.

Now, we come to the proof of the implication $(ii) \Rightarrow (i)$ in Theorem 21. Assume, if possible, that (i) equivalently (iv) as above is false, i.e., the set of semistable points of $(G/B)^s$ is empty (and (ii) is true). Thus, any point $x = (\bar{g}_1, \ldots, \bar{g}_s) \in (G/B)^s$ is unstable. Choose a generic x. Let $\lambda = (\delta, a), P, \gamma, f, w_j$ be as above. It follows from Theorem 26 that $\bigcap_{j=1}^s g_j B w_j P \subset G/P$ is the single point f and, since x is generic, we get

$$[\bar{\Lambda}_{w_1}^P] \cdot \cdots \cdot [\bar{\Lambda}_{w_s}^P] = [\bar{\Lambda}_e^P] \in H^*(G/P, \mathbb{Z}). \tag{31}$$

We now claim that the s-tuple $(w_1, \cdots, w_s) \in (W/W_P)^s$ is L-movable.

Write $g_j = f p_j w_j^{-1} b_j$, for some $p_j \in P(\gamma)$ and $b_j \in B$. Hence,

$$\delta(t)\bar{g}_j = f\gamma(t) p_j w_j^{-1} B = f\gamma(t) p_j \gamma^{-1}(t) w_j^{-1} B \in G/B.$$

Define,

$$l_j = \lim_{t \to 0} \gamma(t) p_j \gamma^{-1}(t).$$

Then, $l_j \in L(\gamma)$. Therefore,

$$\lim_{t \to 0} \delta(t) x = (f l_1 w_1^{-1} B, \ldots, f l_s w_s^{-1} B).$$

By Theorem 24, $\lambda \in \Lambda(\lim_{t \to 0} \delta(t) x)$. We note that, for $j = 1, \ldots, s$,

$$fP(\gamma) \in (f l_j w_j^{-1}) B w_j P(\gamma).$$

Applying Theorem 26 to the unstable point $x_o = \lim_{t \to 0} \delta(t) x$ yields

(†) fP is the only point in the intersection $\bigcap_{j=1}^s f l_j w_j^{-1} B w_j P$.

Translating by f, we get:

(‡) eP is the only point in the intersection

$$\Omega := \bigcap l_j w_j^{-1} B w_j P.$$

Since the sequence $(w_1, \cdots, w_s)$ satisfies equation (31); in particular, it satisfies equation (7). Therefore, the expected dimension of Ω is 0 and so is its actual dimension by (‡). If this intersection Ω is not transverse at e, then by intersection theory ([Fu1, Remark 8.2]), the local multiplicity at e is > 1, each $w_j^{-1}Bw_jP$ being smooth.

Further, G/P being a homogenous space, any other component of the intersection

$$\bigcap l_j \overline{w_j^{-1}Bw_jP}.$$

contributes nonnegatively to the intersection product $[\bar{\Lambda}^P_{w_1}] \cdot \cdots \cdot [\bar{\Lambda}^P_{w_s}]$ (cf., [Fu1, §12.2]). Thus, from equation (31), we get that the intersection $\bigcap l_j w_j^{-1}Bw_jP$ is transverse at $e \in G/P$, proving that $(w_1, \ldots, w_s)$ is L-movable. Clearly, by condition (ii), we see that the s-tuple $(w_1, \ldots, w_s)$ satisfies

$$\sum_{j=1}^{s} \nu_j(w_j x_{i_P}) \leq 0.$$

This contradicts the (b)-part of Theorem 26. Thus, the set of semistable points of $(G/B)^s$ is nonempty, proving condition (i) of Theorem 21. $\square$

7.4 Proof of Theorem 26

Even though proof of Theorem 26 can be extracted from [LM], we did not find it explicitly stated there. So, for completeness, we give its proof couched entirely in algebro-geometric language.

Let $x = (g_1B, \cdots, g_sB) \in (G/B)^s$ be any unstable point and let $\lambda = (\delta, a)$ be a Kempf OPS associated to x. Write

$$Y = \frac{X_\delta}{a} = \sum_{\alpha_i \in S_\delta} c_i x_i,$$

where S_δ is a subset of the set of simple roots such that each c_i is nonzero (and positive). Let $\gamma \in O(G)$ and $f \in G$ be such that $f\gamma f^{-1} = \delta$ and $\dot{\gamma} = X_\delta \in \mathfrak{h}_+$. Define $w_j \in W/W_{P(\gamma)}$ by $fP(\gamma) \in g_jBw_jP(\gamma)$ for $j = 1, \ldots, s$.

With this notation, Theorem 26 can equivalently be formulated as the following:

(a)′ For $\alpha_i \in S_\delta$, $fP(\lambda_i)$ is the only point in $\bigcap_j g_j B w_j P(\lambda_i)$, where λ_i is the OPS with $\dot{\lambda}_i = x_i$. Moreover,

$$\sum_{j=1}^{s} \nu_j(w_j x_i) > 0. \tag{32}$$

Note that

$$P(\lambda_i) \supset P(\gamma). \tag{33}$$

We also note the following additional property:

(a_1) $fP(\gamma)$ is the only point in $\bigcap g_j B w_j P(\gamma)$.

For (a_1), let $hP(\gamma)$ be some other point in the intersection. Then, from Lemma 14, we have $\mu^{\mathbb{L}}(x, h\gamma h^{-1}) = \mu^{\mathbb{L}}(x, f\gamma f^{-1})$. Further, clearly $f\gamma f^{-1}$ and $h\gamma h^{-1}$ have the same norm. Therefore, by Theorem 23, $fP(\gamma)f^{-1} = hP(\gamma)h^{-1}$, i.e., $hP(\gamma) = fP(\gamma)$ as elements of $G/P(\gamma)$.

To prove (a)′, it is convenient to prove the following auxiliary lemma.

Lemma 27. *The inequality (32) holds. Moreover, let θ be an OPS such that $X_\theta = x_i$ for $\alpha_i \in S_\delta$ and*

$$\frac{-\mu^{\mathbb{L}}(x,\theta)}{q(\dot{\theta})} \geq \frac{-\mu^{\mathbb{L}}(x, f\lambda_i f^{-1})}{q(x_i)}. \tag{34}$$

Then, equality holds in the above equation (34) and $P(\theta) = P(f\lambda_i f^{-1})$.

This lemma will yield (a)′, because if $hP(\lambda_i)$ were another point of the intersection $\bigcap_j g_j B w_j P(\lambda_i)$, then (by using Lemma 14 again)

$$\mu^{\mathbb{L}}(x, h\lambda_i h^{-1}) = \mu^{\mathbb{L}}(x, f\lambda_i f^{-1})$$

and, $q(h\lambda_i h^{-1}) = q(f\lambda_i f^{-1})$. Thus, from the above lemma, we get that $hP(\lambda_i) = fP(\lambda_i)$. This completes the proof of (a)′.

Proof of Lemma 27: Let $\widetilde{H}$ be a maximal torus of G contained in $P(\theta) \cap P(\delta)$, where $\lambda = (\delta, a)$ is a Kempf's OPS associated to x. Using equation (30) and the conjugacy of maximal tori in the algebraic group $P(\theta)$, replace θ with $p\theta p^{-1}$ for some $p \in P(\theta)$ chosen so that $p\theta p^{-1} \in O(\widetilde{H})$ (and the inequality (34) is satisfied for θ replaced by $p\theta p^{-1}$).

Let $(\widetilde{\delta}, a)$ be a Kempf's OPS (corresponding to the point $x \in \mathbb{X}$) such that $\widetilde{\delta} \in O(\widetilde{H})$. Find $b \in G$ so that $b^{-1}\widetilde{H}b = H$ and $b^{-1}\widetilde{\delta}b = X_{\widetilde{\delta}} = X_\delta = \dot{\gamma}$.

From the uniqueness of the parabolics associated to Kempf's OPS (cf. Theorem 23), we get:

$$bP(\gamma) = fP(\gamma) \in G/P(\gamma). \tag{35}$$

Now, $b^{-1}\theta b \in O(H)$ and $X_\theta = x_i$. Therefore,

$$\mathrm{Ad}(b^{-1})\dot{\theta} = wx_i, \ \text{ for some } w \in W. \tag{36}$$

Let $\mathfrak{h}^{\mathbb{Q}}$ be the $\mathbb{Q}$-vector subspace of $\mathfrak{h}$ spanned by $\dot{\nu}$, where ν runs over $O(H)$, and let $\mathfrak{h}^{\mathbb{Q}}_+ := \mathfrak{h}^{\mathbb{Q}} \cap \mathfrak{h}_+$, where $\mathfrak{h}_+ := \{h \in \mathfrak{h} : \alpha_i(h) \geq 0 \forall \text{ simple roots } \alpha_i\}$ is the set of dominant elements of $\mathfrak{h}$. Define the function $\mathfrak{L} = \mathfrak{L}_{\mathbb{L},x,b} : \mathfrak{h}^{\mathbb{Q}} \to \mathbb{Q}$ as follows. For any $\beta \in O(H)$ and $r \geq 0 \in \mathbb{Q}$,

$$\mathfrak{L}(r\dot{\beta}) = -r\mu^{\mathbb{L}}(x, b\beta b^{-1}).$$

Let V be a finite dimensional representation of G together with a G-equivariant embedding $i : \mathbb{X} \to \mathbb{P}(V)$ such that $i^*(\mathcal{O}(1))$ is G-equivariantly isomorphic with $\mathbb{L}^M$ for some $M > 0$. We can take, e.g., $V = H^0((G/B)^s, \mathbb{L}^M)^*$ for any $M > 0$. Define a twisted action of G on V via

$$g \odot v = (bgb^{-1}) \cdot v, \ \text{ for } g \in G, v \in V.$$

Find a basis $\{e_1, \ldots, e_n\}$ of V so that, under the twisted action, H acts by the character η_l on e_l, i.e., $t \odot e_l = \eta_l(t)e_l$, for $t \in H$. Write $i(x) = [\sum x_l e_l]$. Then, by equation (13), for any $h \in \mathfrak{h}^{\mathbb{Q}}$,

$$\mathfrak{L}(h) = -\frac{1}{M}\max_{l: x_l \neq 0}(-\dot{\eta}_l(h)) = \frac{1}{M}\min_{l: x_l \neq 0}(\dot{\eta}_l(h)).$$

The function $\mathfrak{L}$ satisfies the following properties:

(P_1) $\mathfrak{L}$ is convex: $\mathfrak{L}(ah_1 + bh_2) \geq a\mathfrak{L}(h_1) + b\mathfrak{L}(h_2)$ for $h_1, h_2 \in \mathfrak{h}^{\mathbb{Q}}$ and positive rational numbers a, b.

(P_2) $\mathfrak{L}(h) = \sum_j \nu_j(w_j h)$, for $h \in \mathfrak{h}^{\mathbb{Q}}_+$; in particular, $\mathfrak{L}$ is linear restricted to $\mathfrak{h}^{\mathbb{Q}}_+$.

(P$_3$) From Kempf's theory, the function $J(h) := \frac{\mathfrak{L}(h)}{q(h)}$ on $\mathfrak{h}^{\mathbb{Q}}$ achieves its maximum uniquely at the positive ray through $Y \in \mathfrak{h}_+^{\mathbb{Q}}$, where Y is defined in the beginning of this subsection 7.4.

Fix $h \in \mathfrak{h}^{\mathbb{Q}}$ and consider the function $v \mapsto J(Y + vh)$, for rational $v \geq 0$. Then, by the convexity of $\mathfrak{L}$ as in (P$_1$), $J(Y + vh) \geq \frac{\mathfrak{L}(Y)+v\mathfrak{L}(h)}{q(Y+vh)}$. View the right hand side as a function of v. It clearly takes the maximum value at $v = 0$. Thus, taking its derivative at $v = 0$, we get:

$$\mathfrak{L}(h)q(Y)^2 - \mathfrak{L}(Y)\langle Y, h\rangle \leq 0,$$

or, that

$$J(h) \leq J(Y)\frac{\langle Y, h\rangle}{q(Y)q(h)}. \tag{37}$$

From now on till the end of this proof, we take i such that $\alpha_i \in S_\delta$. We note that $Y + vx_i \in \mathfrak{h}_+$ for small (positive or negative) values of v. Moreover, by (P$_2$), $\mathfrak{L}(Y + vx_i) = \mathfrak{L}(Y) + v\mathfrak{L}(x_i)$ for small values of v. Thus, the function $v \mapsto \frac{\mathfrak{L}(Y)+v\mathfrak{L}(x_i)}{q(Y+vx_i)}$ has a local maximum at 0; in particular, its derivative at $v = 0$ is zero. This gives:

$$J(x_i) = J(Y)\frac{\langle Y, x_i\rangle}{q(Y)q(x_i)}. \tag{38}$$

Moreover, since $\langle x_i, x_j\rangle \geq 0$ for simple roots α_i and α_j and $J(Y) > 0$, we get that $J(x_i) > 0$. That is,

$$\mu^{\mathbb{L}}(x, b\lambda_i b^{-1}) < 0,$$

where $\lambda_i \in O(H)$ is defined by $\dot{\lambda}_i = x_i$. But it follows from equation (35) that

$$\mu^{\mathbb{L}}(x, b\lambda_i b^{-1}) = \mu^{\mathbb{L}}(x, f\lambda_i f^{-1}). \tag{39}$$

We conclude using Lemma 14 that the inequality (32) holds. Our assumption (34) now reads as

$$J(wx_i) \geq J(x_i). \tag{40}$$

But according to the inequality (37),

$$J(wx_i) \leq J(Y)\frac{\langle Y, wx_i\rangle}{q(Y)q(wx_i)}, \tag{41}$$

and by equation (38),

$$J(x_i) = J(Y)\frac{\langle Y, x_i\rangle}{q(Y)q(x_i)}. \tag{42}$$

It is also easy to see that $\langle Y, x_i\rangle > \langle Y, wx_i\rangle$, if $wx_i \neq x_i$. Combining equations (40)–(42), we therefore conclude that $wx_i = x_i$. Thus, by equation (36), $\theta = b\lambda_i b^{-1}$ and hence

$$P(\theta) = P(b\lambda_i b^{-1}) = P(f\lambda_i f^{-1}),$$

by equations (33) and (35). Finally, by equations (36) and (39), the inequality (34) is in fact an equality. This proves Lemma 27. □

7.5 L-movability and the eigenvalue problem

Let G be a connected semisimple group. Choose a maximal compact subgroup K of G with Lie algebra $\mathfrak{k}$. Then, there is a natural homeomorphism $C : \mathfrak{k}/K \to \mathfrak{h}_+$, where K acts on $\mathfrak{k}$ by the adjoint representation.

Now, one of the main aims of the *eigenvalue problem* is to describe the set $\Gamma(s, K) :=$

$$\{(h_1, \dots, h_s) \in \mathfrak{h}_+^s \mid \exists (k_1, \dots, k_s) \in \mathfrak{k}^s \colon \sum_{j=1}^{s} k_j = 0 \text{ and } C(k_j) = h_j \forall j = 1, \dots, s\}.$$

Given a standard maximal parabolic subgroup P, let ω_P denote the corresponding fundamental weight, i.e., $\omega_P(\alpha_i^\vee) = 1$, if $\alpha_i \in \Delta \setminus \Delta(P)$ and 0 otherwise, where $\alpha_i^\vee$ is the fundamental coroot corresponding to the simple root α_i. This is invariant under the Weyl group W_P of P.

The following theorem is one of our main results which gives a solution of the eigenvalue problem.

Theorem 28. *Let $(h_1, \dots, h_s) \in \mathfrak{h}_+^s$. Then, the following are equivalent:*

(a) $(h_1, \dots, h_s) \in \Gamma(s, K)$.

(b) For every standard maximal parabolic subgroup P in G and every choice of s-tuples $(w_1, \cdots, w_s) \in (W^P)^s$ such that

$$[\bar{\Lambda}_{w_1}^P] \odot_0 \cdots \odot_0 [\bar{\Lambda}_{w_s}^P] = [\bar{\Lambda}_e^P] \in \big(H^*(G/P, \mathbb{Z}), \odot_0\big),$$

the following inequality holds:

$$\omega_P(\sum_{j=1}^{s} w_j^{-1} h_j) \leq 0.$$

Proof. Observe first that, under the identification of $\mathfrak{h}$ with $\mathfrak{h}^*$ induced from the Killing form, $\mathfrak{h}_+$ is isomorphic with the set D of dominant weights of $\mathfrak{h}^*$. In fact, under this identification, x_i corresponds with $2\omega_i/\langle\alpha_i, \alpha_i\rangle$, where ω_i denotes the i-th fundamental weight. Let $D_{\mathbb{Z}}$ be the set of dominant integral weights. Define

$$\begin{aligned}\bar{\Gamma}(s) := \{&(\nu_1, \cdots, \nu_s) \in D^s : N\nu_j \in D_{\mathbb{Z}} \text{ for all } j \text{ and} \\ &H^0\big((G/B)^s, \mathcal{L}(N\nu_1) \boxtimes \cdots \boxtimes \mathcal{L}(N\nu_s)\big)^G \neq 0 \text{ for some } N > 0\}.\end{aligned}$$

Then, under the identification of $\mathfrak{h}_+$ with D (and hence of $\mathfrak{h}_+^s$ with D^s), $\Gamma(s, K)$ corresponds to the closure of $\bar{\Gamma}(s)$. In fact, $\bar{\Gamma}(s)$ consists of the rational points of the image of $\Gamma(s, K)$ (cf., e.g., [Sj, Theorem 7.6]). Since x_i corresponds with $2\omega_i/\langle\alpha_i, \alpha_i\rangle$, the theorem follows from Theorem 21.

□

8 Nonvanishing of Products in the Cohomology of Flag Varieties (Horn Inequalities)

We give two inductive criteria (actually, only necessary conditions) to determine when the product of a number of Schubert cohomology classes of G/P is nonzero. The first criterion (Theorem 29) is in terms of the characters, whereas the second one (Theorem 36) is in terms of dimension counts.

Theorem 29. *Assume that $(w_1, \cdots, w_s) \in (W^P)^s$ satisfies equation (7) and that $[\bar{\Lambda}_{w_1}^P] \cdot \cdots \cdot [\bar{\Lambda}_{w_s}^P] = d[\bar{\Lambda}_1^P]$ in $H^*(G/P)$, for some nonzero d. Then,*

(a) For each $\alpha_i \in \Delta \setminus \Delta(P)$, the following inequality holds:

$$\big((\sum_{j=1}^{s} \chi_{w_j}) - \chi_1\big)(x_i) \leq 0, \tag{43}$$

where χ_w is defined in Definition 5.

(b) For any standard parabolic Q_L of L (i.e., $Q_L \supset B_L$), and $u_1, \cdots, u_s \in W_L/W_{Q_L}$ such that

$$[\bar{\Lambda}_{u_1}^{Q_L}] \cdot \cdots \cdot [\bar{\Lambda}_{u_s}^{Q_L}] \neq 0 \in H^*(L/Q_L),$$

the inequality

$$\sum_{j=1}^{s} \chi_{w_j}(u_j x_p) \leq \chi_1(x_p) \tag{44}$$

holds for any p such that $\alpha_p \in \Delta(P) \setminus \Delta(Q_L)$.

Proof. Let $\nu_1, \cdots, \nu_s \in X(H)$ and let

$$0 \neq \sigma \in H^0\left((P/B_L)^s, \mathcal{L}(\nu_1) \boxtimes \cdots \boxtimes \mathcal{L}(\nu_s)\right)^P.$$

Assume that σ does not vanish at $(\bar{p}_1, \cdots, \bar{p}_s) \in (P/B_L)^s$. Then, for every admissible OPS $\lambda \in O(P)$, we have the following inequality obtained from Proposition 10(b) and Lemma 14:

$$\nu_1([\bar{p}_1, \lambda] X_\lambda) + \cdots + \nu_s([\bar{p}_s, \lambda] X_\lambda) \leq 0, \tag{45}$$

where X_λ is defined by equation (16).

Now, Theorem 29 follows immediately from the following proposition together with Proposition 2 and Lemma 7. (Observe that χ_1 is fixed by W_L.)

□

Proposition 30. *Let $\nu_1, \cdots, \nu_s \in X(H)$ and let*

$$H^0\left((P/B_L)^s, \mathcal{L}(\nu_1) \boxtimes \cdots \boxtimes \mathcal{L}(\nu_s)\right)^P \neq 0. \tag{46}$$

Then,

(a) For each $\alpha_i \in \Delta \setminus \Delta(P)$, we have $\sum_{j=1}^{s} \nu_j(x_i) \leq 0$.

(b) For any standard parabolic subgroup Q_L of L and $u_1, \cdots, u_s \in W_L/W_{Q_L}$ such that $[\bar{\Lambda}_{u_1}^{Q_L}] \cdot \cdots \cdot [\bar{\Lambda}_{u_s}^{Q_L}] \neq 0 \in H^*(L/Q_L)$, we have:

$$\sum_{j=1}^{s} \nu_j(u_j x_p) \leq 0, \text{ for all } \alpha_p \in \Delta(P) \setminus \Delta(Q_L).$$

Proof. For (a), apply equation (45) to the admissible OPS $\lambda = t^{x_i}$, which is central in L. In this case, by equation (15), $[\bar{p}, \lambda] = 1$ for any $\bar{p} \in P/B_L$ and $X_\lambda = \dot{\lambda} = x_i$.

To prove (b), pick a nonzero σ in the vector space (46). Let $Z \subset L^s$ be the set of points $(l_1, \cdots, l_s)$ such that

$$l_1 B_L u_1 Q_L \cap \cdots \cap l_s B_L u_s Q_L \neq \emptyset.$$

By the assumption in (b), Z is nonempty (and open). Let Z_σ be the subset of P^s consisting of $(p_1, \cdots, p_s)$ such that σ does not vanish at $(\bar{p}_1, \cdots, \bar{p}_s) \in (P/B_L)^s$, where $\bar{p}_j := p_j B_L$. Since $\sigma \neq 0$, Z_σ is a nonempty open subset of P^s. Consider the projection $\pi : P^s \to L^s$ under the decomposition $P = U \cdot L$ and pick

$$(p_1, \cdots, p_s) \in Z_\sigma \cap \pi^{-1}(Z).$$

Since Z and Z_σ are nonempty Zariski open subsets, the intersection $Z_\sigma \cap \pi^{-1}(Z)$ is nonempty and open.

Let $p_j = u_j l_j$, for $j = 1, \cdots, s$, where $u_j \in U$ and $l_j \in L$. Pick

$$l \in l_1 B_L u_1 Q_L \cap \cdots \cap l_s B_L u_s Q_L.$$

Consider the admissible OPS $\lambda(t) = l t^{x_p} l^{-1}$, for $\alpha_p \in \Delta(P) \setminus \Delta(Q_L)$. (To prove that this is admissible, use Lemma 12.) Since λ is conjugate to the OPS $\lambda_o := t^{x_p}$ lying in H and, moreover, $x_p \in \mathfrak{h}$ is L-dominant, we get $X_\lambda = x_p$. Clearly, $[\bar{p}_j, \lambda] = u_j$, for any $j = 1, \cdots, s$. We can therefore use equation (45) to conclude that

$$\sum_{j=1}^{s} \nu_j(u_j x_p) \leq 0.$$

This proves (b).

$\square$

Remark 31. (a) Let $\nu_1, \cdots, \nu_s \in X(H)$ and let $\mathbb{L}$ be the line bundle $\mathcal{L}(\nu_1) \boxtimes \cdots \boxtimes \mathcal{L}(\nu_s)$ on $(P/B_L)^s$. Assume that for every $\alpha_i \in \Delta \setminus \Delta(P)$, we have $\sum_{j=1}^{s} \nu_j(x_i) = 0$. Then, the restriction map

$$H^0\left((P/B_L)^s, \mathbb{L}\right)^P \to H^0\left((L/B_L)^s, \mathbb{L}\right)^L$$

is an isomorphism. If the above equality is violated for some $\alpha_i \in \Delta \setminus \Delta(P)$, then $H^0\left((L/B_L)^s, \mathbb{L}\right)^L = 0$.

This is proved by the same technique that was used in the proof of Theorem 15 (cf., Section 5).

(b) In Theorem 29, the validity of property (b) for every standard parabolic subgroup Q_L of L is equivalent to the corresponding property only for the standard *maximal* parabolic subgroups Q_L of L. This can be proved by the same technique as developed in Section 7.

In the L-movable case, we have the following refinement of Theorem 29. To state this refinement, we need the following notation.

For any $w \in W^P$ and a central character c of L (i.e., an algebraic group homomorphism $Z(L) \to \mathbb{G}_m$, $Z(L) \subset H$ being the center of the Levi subgroup L of P), define

$$\chi_w^c = \sum_{\beta \in R(w,c)} \beta, \tag{47}$$

where

$$R(w,c) := \{\beta \in \left(R^+ \setminus R_{\mathfrak{l}}^+\right) \cap w^{-1}R^+ : e^{\beta}_{|Z(L)} = c\}. \tag{48}$$

Observe that

$$\chi_w = \sum_c \chi_w^c, \tag{49}$$

where the sum runs over all the central characters of L such that $\chi_1^c \neq 0$.

Theorem 32. *Assume that the s-tuple $(w_1, \cdots, w_s) \in (W^P)^s$ is L-movable. Then,*

1. *For any central character c of L such that $\chi_1^c \neq 0$, we have*

$$\sum_{j=1}^{s} \mid R(w_j, c) \mid = \mid R(1,c) \mid, \tag{50}$$

   where $|\cdot|$ denotes the cardinality of the enclosed set.

2. *For any standard parabolic Q_L of L and $u_1, \cdots, u_s \in W_L/W_{Q_L}$ such that*

$$[\bar{\Lambda}_{u_1}^{Q_L}] \cdot \cdots \cdot [\bar{\Lambda}_{u_s}^{Q_L}] \neq 0 \in H^*(L/Q_L),$$

   and any central character c of L such that $\chi_1^c \neq 0$, the following inequality is satisfied for any $\alpha_p \in \Delta(P) \setminus \Delta(Q_L)$:

$$\sum_{j=1}^{s} \chi_{w_j}^c(u_j x_p) \leq \chi_1^c(x_p). \tag{51}$$

Remark 33. (a) Observe that in the L-movable case, by virtue of Theorem 15, the inequality (43) is, in fact, an equality.

(b) The inequalities (51) summed over all the central characters c of L such that $\chi_1^c \neq 0$ is nothing but the inequality (44) (use the identity (49)). Thus, Theorem 32 is a refinement of Theorem 29 in the L-movable case.

Proof. (of Theorem 32) For any central character c of L and $w \in W^P$, let T_w^c be the B_L-submodule of T_w defined by

$$T_w^c := \{v \in T_w : t \cdot v = c(t)v, \forall\, t \in Z(L) \subset B_L\}.$$

This gives rise to the L-equivariant vector bundle on L/B_L:

$$\mathcal{T}_w^c := L \times_{B_L} T_w^c.$$

Similarly, we can define the B_L-submodule T^c of T and the associated L-equivariant vector bundle $\mathcal{T}^c$ on L/B_L and $\mathcal{T}_s^c$ on $(L/B_L)^s$. Analogous to Lemma 6, we have:

$$\det(\mathcal{T}^c/\mathcal{T}_w^c) = \mathcal{L}(\chi_w^c), \tag{52}$$

as L-equivariant vector bundles on L/B_L. Let Θ_o denote the restriction of the bundle map Θ (defined by (10)) to the subvariety $(L/B_L)^s \subset (P/B_L)^s$. Then, by Corollary 8, Θ_o is an isomorphism over a dense open subset of $(L/B_L)^s$. From this we see that, for any central character c of L such that $\chi_1^c \neq 0$,

$$\Theta_{o|\mathcal{T}_s^c} : \mathcal{T}_s^c \to \oplus_{j=1}^s \pi_j^*(\mathcal{T}^c/\mathcal{T}_{w_j}^c) \tag{53}$$

is an isomorphism over a dense open subset of $(L/B_L)^s$, where $\pi_j : (L/B_L)^s \to L/B_L$ is the projection onto the j-th factor. Therefore, the ranks of the two sides of equation (53) coincide. This gives the equality (50).

Taking the determinant of $\Theta_{o|\mathcal{T}_s^c}$ and using the equation (52), we get a nonzero bundle map

$$\theta_o^c : \det(\mathcal{T}_s^c) \to \mathcal{L}(\chi_{w_1}^c) \boxtimes \cdots \boxtimes \mathcal{L}(\chi_{w_s}^c),$$

i.e., a nonzero section of the line bundle

$$\mathcal{L}(\chi_{w_1}^c - \chi_1^c) \boxtimes \cdots \boxtimes \mathcal{L}(\chi_{w_s}^c)$$

on $(L/B_L)^s$. Now, applying Proposition 30 for the case $G = L, P = L$, and observing that χ_1^c is fixed by W_L, we get the theorem. $\square$

Question 34. We would like to ask if the converse of Theorem 29 is true. Specifically, take $(w_1, \cdots, w_s) \in (W^P)^s$ satisfying the equation (7) and assume that the conditions (a) and (b) of Theorem 29 are satisfied. Then, is it true that

$$[\bar{\Lambda}^P_{w_1}] \cdot \cdots \cdot [\bar{\Lambda}^P_{w_s}] = d[\bar{\Lambda}^P_1]$$

in $H^*(G/P)$, for some nonzero d. For $G = SL_n$ and any maximal parabolic subgroup P, this question has an affirmative answer [Fu2].

One could ask the corresponding (weaker) question for L-movable s-tuples, i.e., is the converse of Theorem 32 true? It may be remarked here that this question (in the L-movable case) for G/B has an affirmative answer by virtue of Corollary 43.

We now come to our second criterion (Theorem 36) to determine when the products of cohomology classes in G/P are nonzero. This criterion is in terms of inequalities involving certain dimension counts.

Let Q be a standard parabolic subgroup of G contained in P and let $Q_L := Q \cap L$ be the associated parabolic in L. Let $\hat{Q}$ be a standard parabolic in G containing Q. Then, we have the canonical identification $L/Q_L \simeq P/Q$ and the standard projection $\tau : P/Q \to G/\hat{Q}$.

Lemma 35. *For any $w \in W^P, v \in W_L, p \in P$ and $b \in B$, we have (setting $g^{-1} = bwp$):*

$$\tau(p^{-1}BvQ) \subset gB(wv)\hat{Q}.$$

Proof. Since $w \in W^P$, by equation (3), we have $wB_Lw^{-1} \subset B$. Further,

$$\tau(p^{-1}BvQ) = \tau(p^{-1}B_LUvQ) = \tau(p^{-1}B_LvQ) = gbwB_Lv\hat{Q} = gbwB_Lw^{-1}(wv)\hat{Q}.$$

We therefore have

$$\tau(p^{-1}BvQ) \subset gbBwv\hat{Q} = gB(wv)\hat{Q}.$$

□

Theorem 36. *Let $w_1, \ldots, w_s \in W^P$ be such that*

$$[\bar{\Lambda}^P_{w_1}] \cdot \cdots \cdot [\bar{\Lambda}^P_{w_s}] \neq 0 \tag{54}$$

in $H^*(G/P, \mathbb{Z})$. Then given the data: $Q \subset P$ a parabolic, $\hat{Q} \subset G$ a parabolic containing Q, elements $u_1, \ldots, u_s \in W_L/W_{Q_L}$ such that

$$[\bar{\Lambda}^{Q_L}_{u_1}] \cdot \cdots \cdot [\bar{\Lambda}^{Q_L}_{u_s}] \neq 0 \tag{55}$$

in $H^*(L/Q_L, \mathbb{Z})$, the following hold (with $\hat{w}_j := w_j u_j$):

(a) $[\bar{\Lambda}^{\hat{Q}}_{\hat{w}_1}] \cdot \dots \cdot [\bar{\Lambda}^{\hat{Q}}_{\hat{w}_s}] \neq 0$ in $H^*(G/\hat{Q}, \mathbb{Z})$. In particular,

$$\sum_j \operatorname{codim}(\Lambda^{\hat{Q}}_{\hat{w}_j}; G/\hat{Q}) \leq \dim(G/\hat{Q}). \tag{56}$$

(b) If $\hat{Q} \cap P = Q$ then,

$$\dim(G/\hat{Q}) - \sum_j \operatorname{codim}(\Lambda^{\hat{Q}}_{\hat{w}_j}; G/\hat{Q}) \geq \dim(L/Q_L) - \sum_j \operatorname{codim}(\Lambda^{Q_L}_{u_j}; L/Q_L). \tag{57}$$

The above inequality (57) is equivalent to the following inequality (in the case $\hat{Q} \cap P = Q$):

$$\mid R(\mathfrak{u}_{\hat{Q}}) \cap R(\mathfrak{u}_P) \mid \geq \sum_{j=1}^{s} \mid R(\mathfrak{u}_{\hat{Q}}) \cap R(\mathfrak{u}_P) \cap \hat{w}_j^{-1} R^+ \mid, \tag{58}$$

where $\mid \cdot \mid$ denotes the cardinality of the enclosed set.

Proof. Take generic g_j for $j = 1, \dots, s$ so that for each standard parabolic $\widetilde{P}$ in G and any $(z_1, \dots, z_s) \in W^s$, the intersection

$$g_1 B z_1 \widetilde{P} \cap \dots \cap g_s B z_s \widetilde{P}$$

is transverse (possibly empty).

We may further assume, by left multiplying each g_j by the same element, that for the given s-tuple $(w_1, \dots, w_s) \in (W^P)^s$,

$$e \in g_1 B w_1 P \cap \dots \cap g_s B w_s P.$$

Choose $p_j \in P, b_j \in B$ such that $e = g_j b_j w_j p_j$, for $j = 1, \dots, s$. Now, consider the intersection in $L/Q_L \simeq P/Q$:

$$\Omega = \overline{p_1^{-1} B u_1 Q} \cap \dots \cap \overline{p_s^{-1} B u_s Q}.$$

Then, Ω is nonempty because of the assumption (55). Each irreducible component of Ω is of dimension at least

$$\dim(L/Q_L) - \sum_{j=1}^{s} \operatorname{codim}(\Lambda^{Q_L}_{u_j}; L/Q_L). \tag{59}$$

By Lemma 35, $\tau(p_j^{-1}Bu_jQ) \subset g_jBw_ju_j\hat{Q} \subset \overline{g_jB\hat{w}_j\hat{Q}}$, for $j = 1, \ldots, s$, under the projection $\tau : P/Q \to G/\hat{Q}$ (which is an embedding if $\hat{Q} \cap P = Q$). So,

$$\tau(\Omega) \subset \bigcap_{j=1}^{s} \overline{g_jB\hat{w}_j\hat{Q}}. \tag{60}$$

The intersection $\Omega' := \bigcap_{j=1}^{s} \overline{g_jB\hat{w}_j\hat{Q}}$ is therefore nonempty (and proper by the choice of g_j). This gives the part (a) of the theorem.

In the case $\hat{Q} \cap P = Q$, τ is an embedding. Comparison of equations (59) and (60) gives the inequality (57) of (b). The inequality (58) follows from (57) and the following Lemma 39. □

Remark 37. This theorem is of interest even in the case $P = B$. In this case, $Q = B$ and any standard parabolic is allowed for $\hat{Q}$.

We recall the following simple lemma.

Lemma 38. *For $w \in W$ (not necessarily in W^P), we have:*

$$\operatorname{codim}(\Lambda_w^P; G/P) = \mid R(\mathfrak{u}_P) \cap (w^{-1}R^+) \mid . \tag{61}$$

Lemma 39. *Let $w \in W^P$, $u \in W_P$ and $\hat{w} := wu$. Let $Q \subset P$ be a standard parabolic and let $\hat{Q}$ be a parabolic in G such that $\hat{Q} \cap P = Q$. Then,*

$$\operatorname{codim}(\Lambda_{\hat{w}}^{\hat{Q}}; G/\hat{Q}) - \operatorname{codim}(\Lambda_u^{Q_L}; L/Q_L) = \mid R(\mathfrak{u}_{\hat{Q}}) \cap R(\mathfrak{u}_P) \cap \hat{w}^{-1}R^+ \mid . \tag{62}$$

Proof. Write R^+ as a disjoint union of three parts (note that since $w \in W^P$, $wR_{\mathfrak{l}}^+ \subset R^+$ by (3))

$$R^+ = wR_{\mathfrak{l}}^+ \sqcup \big(wR(\mathfrak{u}_P^-) \cap R^+\big) \sqcup \big(wR(\mathfrak{u}_P) \cap R^+\big),$$

where $\mathfrak{u}_P^-$ is the opposite nil-radical of $\mathfrak{p}$. Take $\hat{w}^{-1} = u^{-1}w^{-1}$ of this decomposition and intersect with $R(\mathfrak{u}_{\hat{Q}})$. The first piece is

$$R(\mathfrak{u}_{\hat{Q}}) \cap u^{-1}R_{\mathfrak{l}}^+ = R(\mathfrak{u}_Q) \cap u^{-1}R_{\mathfrak{l}}^+, \text{ since } \hat{Q} \cap P = Q.$$

Thus, by Lemma 38,

$$\mid R(\mathfrak{u}_{\hat{Q}}) \cap u^{-1}R_{\mathfrak{l}}^+ \mid = \operatorname{codim}(\Lambda_u^{Q_L}; L/Q_L). \tag{63}$$

The second piece is

$$R(\mathfrak{u}_{\hat{Q}}) \cap u^{-1}R(\mathfrak{u}_P^-) \cap u^{-1}w^{-1}R^+.$$

But $u^{-1}R(\mathfrak{u}_P^-) = R(\mathfrak{u}_P^-)$ and clearly

$$R(\mathfrak{u}_{\hat{Q}}) \cap R(\mathfrak{u}_P^-) = \emptyset. \tag{64}$$

So the second piece gives us the empty set.

The third piece is

$$R(\mathfrak{u}_{\hat{Q}}) \cap u^{-1}R(\mathfrak{u}_P) \cap u^{-1}w^{-1}R^+ = R(\mathfrak{u}_{\hat{Q}}) \cap R(\mathfrak{u}_P) \cap u^{-1}w^{-1}R^+. \tag{65}$$

Finally, by Lemma 38,

$$\mid R(\mathfrak{u}_{\hat{Q}}) \cap u^{-1}w^{-1}R^+ \mid = \operatorname{codim}(\Lambda_{\hat{w}}^{\hat{Q}}; G/\hat{Q}). \tag{66}$$

Combining the equations (63)–(66), we get equation (62), proving the lemma. $\square$

Remark 40. (a) It is easy to see that in the case when $\hat{Q}$ is a minuscule maximal parabolic subgroup and $Q \neq P$ (and $\hat{Q} \cap P = Q$), the inequality (58) is the same as the inequality (44).

Also, in the case $\hat{Q} = Q$ and P is a minuscule maximal parabolic subgroup, the inequality (58) is the same as the inequality (43).

It may be remarked that, in general, the inequality (44) is a certain 'weighted' version of the inequality (58).

(b) In the general case, we do not know if the system of inequalities (43) and (44) is equivalent to the system of inequalities (58). In fact, we would expect that the former set is more refined than the latter.

(c) In his PhD thesis "Vanishing and non-vanishing criteria for branching Schubert calculus," Kevin Purbhoo has given some criteria for determining which of the Schubert intersection numbers vanish in terms of a combinatorial recipe which he calls 'root game.'

9 A further study of $\big(H^*(G/P, \mathbb{C}), \odot_0\big)$

For any Lie algebra $\mathfrak{s}$ and a subalgebra $\mathfrak{t}$, let $H^*(\mathfrak{s}, \mathfrak{t})$ be the Lie algebra cohomology of the pair $(\mathfrak{s}, \mathfrak{t})$ with trivial coefficients. Recall (cf. [Ku2, §3.1])

that this is the cohomology of the cochain complex

$$C^\bullet(\mathfrak{s},\mathfrak{t}) = \{C^p(\mathfrak{s},\mathfrak{t})\}_{p\geq 0}, \quad \text{where}$$
$$C^p(\mathfrak{s},\mathfrak{t}) := \mathrm{Hom}_{\mathfrak{t}}\big(\wedge^p(\mathfrak{s}/\mathfrak{t}),\mathbb{C}\big).$$

We now return to the notation of §2. For any (positive) root $\beta \in R^+$, let $y_\beta \in \mathfrak{g}_\beta$ be the corresponding Chevalley root vector (which is unique up to a sign) and let $y_{-\beta} \in \mathfrak{g}_{-\beta}$ be the vector such that $\langle y_\beta, y_{-\beta}\rangle = 1$ under the Killing form. For any $w \in W^P$, let $\Phi_w := w^{-1}R^- \cap R^+ \subset R(\mathfrak{u}_P)$. Then,

$$\sum_{\beta\in\Phi_w} \beta = \rho - w^{-1}\rho. \tag{67}$$

In particular, $\Phi_v = \Phi_w$ iff $v = w$. Let $\Phi_w = \{\beta_1, \cdots, \beta_p\} \subset R(\mathfrak{u}_P)$. Set $y_w := y_{\beta_1} \wedge \cdots \wedge y_{\beta_p} \in \wedge^p(\mathfrak{u}_P)$, determined up to a sign. Then, up to scalar multiples, y_w is the unique weight vector of $\wedge(\mathfrak{u}_P)$ with weight $\rho - w^{-1}\rho$. Similarly, we can define $y_w^- \in \wedge^p(\mathfrak{u}_P^-)$ of weight $w^{-1}\rho - \rho$.

We recall the following fundamental result due to Kostant [Ko1].

Theorem 41. *For any standard parabolic subgroup P of G,*

$$H^p(\mathfrak{u}_P) = \bigoplus_{\substack{w\in W^P:\\ \ell(w)=p}} M_w,$$

as $\mathfrak{l}$-modules, where M_w is the unique irreducible $\mathfrak{l}$-submodule of $H^p(\mathfrak{u}_P)$ with highest weight $w^{-1}\rho - \rho$ (which is $\mathfrak{l}$-dominant for any $w \in W^P$). This has a highest weight vector $\phi_w \in \wedge^p(\mathfrak{u}_P)^*$ defined by $\phi_w(y_w) = 1$ and $\phi_w(y) = 0$ for any weight vector of $\wedge^p(\mathfrak{u}_P)$ of weight $\neq \rho - w^{-1}\rho$.

Similarly, for the opposite nil-radical $\mathfrak{u}_P^-$,

$$H^p(\mathfrak{u}_P^-) = \bigoplus_{\substack{w\in W^P:\\ \ell(w)=p}} N_w,$$

as $\mathfrak{l}$-modules, where N_w is the unique irreducible $\mathfrak{l}$-submodule of $H^p(\mathfrak{u}_P^-)$ isomorphic with the dual M_w^* and it has a lowest weight vector $\phi_w^- \in \wedge^p(\mathfrak{u}_P^-)^*$ defined by $\phi_w^-(y_w^-) = 1$ and $\phi_w^-(y) = 0$ for any weight vector of $\wedge^p(\mathfrak{u}_P^-)$ of weight $\neq w^{-1}\rho - \rho$.

Thus,

$$[H^p(\mathfrak{u}_P) \otimes H^q(\mathfrak{u}_P^-)]^{\mathfrak{l}} = 0, \quad \textit{unless } p = q, \textit{ and}$$
$$[H^p(\mathfrak{u}_P) \otimes H^p(\mathfrak{u}_P^-)]^{\mathfrak{l}} \simeq \bigoplus_{\substack{w \in W^P, \\ \ell(w)=p}} \mathbb{C}\xi^w,$$

where $\xi^w \in [M_w \otimes N_w]^{\mathfrak{l}}$ is the unique element whose H-equivariant projection to $(M_w)_{w^{-1}\rho-\rho} \otimes N_w$ is the element $\phi_w \otimes \phi_w^-$, $(M_w)_{w^{-1}\rho-\rho}$ being the weight space of M_w corresponding to the weight $w^{-1}\rho - \rho$. (Observe that the ambiguity in the sign of y_w disappears in the definition of ξ^w giving rise to a completely unique element.)

Theorem 42. *For any standard parabolic subgroup P of G, there is a graded algebra isomorphism*

$$\phi : \big(H^*(G/P, \mathbb{C}), \odot_0\big) \simeq \big[H^*(\mathfrak{u}_P) \otimes H^*(\mathfrak{u}_P^-)\big]^{\mathfrak{l}}$$

such that

$$\phi\big([\bar{\Lambda}_w^P]\big) = \left(\frac{i}{2\pi}\right)^{\dim G/P - \ell(w)} \xi^{w_o w w_o^P} \prod_{\alpha \in \Phi_{w_o w w_0^P}} \langle \rho, \alpha \rangle, \tag{68}$$

where $H^*(G/P, \mathbb{C})$ is equipped with the product $\odot_0$ as in Definition 18, and we take the tensor product algebra structure on the right side.

Proof. Let $x_P := \sum_{\alpha_i \in \Delta \setminus \Delta(P)} x_i$. Consider the graded, multiplicative decreasing filtration $\mathcal{A} = \{\mathcal{A}_m\}_{m \geq 0}$ of $H^*(G/P, \mathbb{C})$ under the intersection product defined as follows:

$$\mathcal{A}_m := \bigoplus_{w \in W_m^P} \mathbb{C}[\bar{\Lambda}_w^P],$$

where $W_m^P := \{w \in W^P : \chi_w(x_P) \geq m\}$. Observe that, by equation (9), $\chi_w(x_P) = \rho + w^{-1}\rho(x_P) \in \mathbb{Z}_+$. Thus, by Proposition 18(a), $\mathcal{A}_m \mathcal{A}_n \subset \mathcal{A}_{m+n}$, i.e., $\mathcal{A}$ respects the algebra structure.

Let $\mathrm{gr}(\mathcal{A}) := \bigoplus_{m \geq 0} \frac{\mathcal{A}_m}{\mathcal{A}_{m+1}}$ be the associated 'gr' algebra. Then $\mathrm{gr}(\mathcal{A})$ acquires two gradings: the first one the 'homogeneous grading' m assigned to the elements of $\frac{\mathcal{A}_m}{\mathcal{A}_{m+1}}$ and the second one the 'cohomological grading' coming from the cohomological degree in $H^*(G/P, \mathbb{C})$. For example, $[\bar{\Lambda}_w^P]$ has bidegree $(\chi_w(x_P), \dim G/P - \ell(w))$.

By the definition of the product $\odot_0$ and Proposition 18(a), the linear map

$$\varphi : \big(H^*(G/P, \mathbb{C}), \odot_0\big) \to \mathrm{gr}(\mathcal{A}), [\bar{\Lambda}^P_w] \mapsto [\bar{\Lambda}^P_w] \mod \mathcal{A}_{\chi_w(x_P)+1},$$

is, in fact, a graded algebra isomorphism with respect to the cohomological grading on $\mathrm{gr}(\mathcal{A})$.

We next introduce another filtration $\{\bar{\mathcal{F}}_m\}_{m\geq 0}$ of $H^*(G/P, \mathbb{C})$ in terms of the Lie algebra cohomology. Recall that choosing a maximal compact subgroup K of G, we can identify $H^*(G/P, \mathbb{C})$ with the cohomology of the K-invariant forms on G/P under the de Rham differential, i.e., with the Lie algebra cohomology $H^*(\mathfrak{g}, \mathfrak{l})$. The underlying cochain complex $C^\bullet = C^\bullet(\mathfrak{g}, \mathfrak{l})$ for $H^*(\mathfrak{g}, \mathfrak{l})$ can be rewritten as

$$C^\bullet := [\wedge^\bullet(\mathfrak{g}/\mathfrak{l})^*]^{\mathfrak{l}} = \mathrm{Hom}_{\mathfrak{l}}\big(\wedge^\bullet(\mathfrak{u}_P) \otimes \wedge^\bullet(\mathfrak{u}_P^-), \mathbb{C}\big).$$

Define a decreasing filtration $\mathcal{F} = \{\mathcal{F}_m\}_{m\geq 0}$ of the cochain complex $C^\bullet$ by subcomplexes:

$$\mathcal{F}_m := \mathrm{Hom}_{\mathfrak{l}}\left(\frac{\wedge^\bullet(\mathfrak{u}_P) \otimes \wedge^\bullet(\mathfrak{u}_P^-)}{\bigoplus_{s+t\leq m-1} \wedge^\bullet_{(s)}(\mathfrak{u}_P) \otimes \wedge^\bullet_{(t)}(\mathfrak{u}_P^-)}, \mathbb{C}\right),$$

where $\wedge^\bullet_{(s)}(\mathfrak{u}_P)$ (resp. $\wedge^\bullet_{(s)}(\mathfrak{u}_P^-)$) denotes the subspace of $\wedge^\bullet(\mathfrak{u}_P)$ (resp. $\wedge^\bullet(\mathfrak{u}_P^-)$) spanned by the $\mathfrak{h}$-weight vectors of weight β with *P-relative height*

$$\mathrm{ht}_P(\beta) :=\mid \beta(x_P) \mid= s.$$

Now, define the filtration $\bar{\mathcal{F}} = \{\bar{\mathcal{F}}_m\}_{m\geq 0}$ of $H^*(\mathfrak{g}, \mathfrak{l}) \simeq H^*(G/P)$ by

$$\bar{\mathcal{F}}_m := \text{Image of } H^*(\mathcal{F}_m) \to H^*(C^\bullet).$$

The filtration $\mathcal{F}$ of $C^\bullet$ gives rise to the cohomology spectral sequence $\{E_r\}_{r\geq 1}$ converging to $H^*(C^\bullet) = H^*(G/P, \mathbb{C})$. By [Ku2, Proof of Proposition 3.2.11], for any $m \geq 0$,

$$E_1^m = \bigoplus_{s+t=m} [H^\bullet_{(s)}(\mathfrak{u}_P) \otimes H^\bullet_{(t)}(\mathfrak{u}_P^-)]^{\mathfrak{l}},$$

where $H^\bullet_{(s)}(\mathfrak{u}_P)$ denotes the cohomology of the subcomplex $(\wedge^\bullet_{(s)}(\mathfrak{u}_P))^*$ of the standard cochain complex $\wedge^\bullet(\mathfrak{u}_P)^*$ associated to the Lie algebra $\mathfrak{u}_P$ and similarly for $H^\bullet_{(t)}(\mathfrak{u}_P^-)$. Moreover, by loc. cit., the spectral sequence degenerates at the E_1 term, i.e.,

$$E_1^m = E_\infty^m. \tag{69}$$

Further, by the definition of P-relative height,

$$[H^\bullet_{(s)}(\mathfrak{u}_P) \otimes H^\bullet_{(t)}(\mathfrak{u}_P^-)]^{\mathfrak{l}} \neq 0 \Rightarrow s = t.$$

Thus,

$$\begin{aligned} E_1^m &= 0, \qquad \text{unless } m \text{ is even and} \\ E_1^{2m} &= [H^\bullet_{(m)}(\mathfrak{u}_P) \otimes H^\bullet_{(m)}(\mathfrak{u}_P^-)]^{\mathfrak{l}}. \end{aligned}$$

In particular, from (69) and the general properties of spectral sequences (cf. [Ku2, Theorem E.9]), we have a canonical algebra isomorphism:

$$\mathrm{gr}(\bar{\mathcal{F}}) \simeq \bigoplus_{m \geq 0} \left[H^\bullet_{(m)}(\mathfrak{u}_P) \otimes H^\bullet_{(m)}(\mathfrak{u}_P^-)\right]^{\mathfrak{l}}, \tag{70}$$

where $\left[H^\bullet_{(m)}(\mathfrak{u}_P) \otimes H^\bullet_{(m)}(\mathfrak{u}_P^-)\right]^{\mathfrak{l}}$ sits inside $\mathrm{gr}(\bar{\mathcal{F}})$ precisely as the homogeneous part of degree $2m$; homogeneous parts of $\mathrm{gr}(\bar{\mathcal{F}})$ of odd degree being zero.

Finally, we claim that, for any $m \geq 0$,

$$\mathcal{A}_m = \bar{\mathcal{F}}_{2m} : \tag{71}$$

Following Kostant [Ko2], take the d-∂ harmonic representative $\hat{s}^w$ in $C^\bullet$ for the cohomology class $[\bar{\Lambda}_w^P]$. An explicit expression is given by [Ko2; Theorem 4.6] together with [KK, Theorem 3.1]. From this explicit expression, we easily see that

$$\mathcal{A}_m \subset \bar{\mathcal{F}}_{2m}. \tag{72}$$

Moreover, from the definition of $\mathcal{A}$, for any $m \geq 0$,

$$\dim \frac{\mathcal{A}_m}{\mathcal{A}_{m+1}} = \#\big\{w \in W^P : \chi_w(x_P) = \rho + w^{-1}\rho(x_P) = m\big\}.$$

Also, by the isomorphism (70) and Theorem 41,

$$\begin{aligned} \dim \frac{\bar{\mathcal{F}}_{2m}}{\bar{\mathcal{F}}_{2m+1}} &= \#\big\{w \in W^P : \rho - w^{-1}\rho(x_P) = m\big\} \\ &= \#\big\{w \in W^P : (\rho - (w_o w w_o^P)^{-1}\rho)(x_P) = m\big\}, \\ &\qquad \text{using the involution } w \mapsto w_o w w_o^P \text{ of } W^P \\ &= \big\{w \in W^P : \rho + w^{-1}\rho(x_P) = m\big\}, \text{ since } w_o^P \text{ keeps } x_P \text{ fixed.} \end{aligned}$$

Thus,

$$\dim \frac{\mathcal{A}_m}{\mathcal{A}_{m+1}} = \dim \frac{\bar{\mathcal{F}}_{2m}}{\bar{\mathcal{F}}_{2m+1}}.$$

Further, for large enough m_o, $\mathcal{A}_{m_o} = \bar{\mathcal{F}}_{2m_o} = 0$. Thus, by decreasing induction on m, we get (71) from (72).

Thus, combining the isomorphisms φ and (70) and using (71), we get the isomorphism ϕ as in the Theorem. The assertion (68) follows from the description of the map φ and the identification (70) together with the explicit expression of the d-∂ harmonic representative $\hat{s}^w$ in $C^\bullet$. This proves the theorem. $\square$

Corollary 43. *The product in* $(H^*(G/B), \odot_0)$ *is given by*

$$\begin{aligned} \epsilon_u^B \odot_0 \epsilon_v^B &= \epsilon_w^B, && \textit{if } \Phi_u \cap \Phi_v = \emptyset \textit{ and } \Phi_w = \Phi_u \sqcup \Phi_v \\ &= 0, && \textit{otherwise.} \end{aligned}$$

Proof. By the above theorem,

$$\phi\big(\epsilon_u^B \odot_0 \epsilon_v^B\big) = \phi\big([\bar{\Lambda}^B_{w_o u}]\big) \cdot \phi\big([\bar{\Lambda}^B_{w_o v}]\big) \tag{73}$$

$$= \Big(\frac{i}{2\pi}\Big)^{\ell(u)+\ell(v)} \Bigg(\prod_{\alpha \in \Phi_u} \langle \rho, \alpha \rangle \cdot \prod_{\beta \in \Phi_v} \langle \rho, \beta \rangle \Bigg) \xi^u \, \xi^v. \tag{74}$$

The right side is clearly 0 if $\Phi_u \cap \Phi_v \neq \emptyset$. So, let us consider the case when $\Phi_u \cap \Phi_v = \emptyset$. In this case, two subcases occur:

1. There exists $w \in W$ such that $\Phi_w = \Phi_u \sqcup \Phi_v$. In particular, $\ell(w) = \ell(u) + \ell(v)$. (Such a w is necessarily unique.)

2. There does not exist any such $w \in W$.

In the first case, $\xi^u \xi^v = \xi^w$ and thus the right side of equation (74) is equal to $\big(\frac{i}{2\pi}\big)^{\ell(w)} \xi^w \prod_{\alpha \in \Phi_w} \langle \rho, \alpha \rangle = \phi(\epsilon_w^B)$. Hence, in this case, $\epsilon_u^B \odot_0 \epsilon_v^B = \epsilon_w^B$.

In the second subcase, by Theorem 41,

$$\xi^u \xi^v = 0, \quad \text{as an element of } [H^*(\mathfrak{u}_B) \otimes H^*(\mathfrak{u}_B^-)]^H.$$

Thus, $\epsilon_u \odot_0 \epsilon_v = 0$. This proves the corollary. $\square$

Remark 44. a) For any $u, v \in W$, $\Phi_u \cap \Phi_v = \emptyset$ iff $\ell(uv^{-1}) = \ell(u) + \ell(v)$.

b) A subset $S \subset R^+$ is called *closed under addition* if for $\alpha, \beta \in S$ such that $\alpha + \beta \in R^+$, we have $\alpha + \beta \in S$. A subset $S \subset R^+$ is called *coclosed under addition* if $R^+ \backslash S$ is closed under addition.

Now, by a result of Kostant [Ko1, Proposition 5.10], a subset S of R^+ is closed and coclosed under addition iff there exists $w \in W$ such that $S = \Phi_w$.

c) For any $u, v, w \in W$, the following two conditions are equivalent:

c$_1$) $\Phi_u \cap \Phi_v = \emptyset$ and $\Phi_w = \Phi_u \sqcup \Phi_v$.

c$_2$) $\ell(w) = \ell(u) + \ell(v)$, $\ell(uv^{-1}) = \ell(u) + \ell(v)$, $\ell(wu^{-1}) = \ell(w) - \ell(u)$ and $\ell(wv^{-1}) = \ell(w) - \ell(v)$.

d) As a consequence of Corollary 43, it is easy to get the following analogue of Chevalley formula:

For a simple reflection s_i,

$$\begin{aligned} \epsilon^B_{s_i} \odot_0 \epsilon^B_v &= \epsilon^B_{vs_i}, && \text{if } v\alpha_i \text{ is a simple root} \\ &= 0, && \text{otherwise.} \end{aligned}$$

Thus, the subalgebra of $\big(H^*(G/B, \mathbb{Z}), \odot_0\big)$ generated by $H^2(G/B, \mathbb{Z})$ is precisely equal to $\bigoplus \mathbb{Z}\epsilon^v$, where the summation runs over those $v \in W$ which can be written as a product of commuting simple reflections with no simple reflection appearing more than once.

10 Tables of the Deformed Product $\odot$ for Rank 3 Groups

We give below the multiplication tables under the deformed product $\odot$ (cf. equation (20)) for G/P for all the rank 3 complex simple groups G and maximal parabolic subgroups P. We will freely follow the convention as in [KLM] without explanation. Since we are only considering maximal parabolics, we have only one indeterminate, which we denote by τ. In the case of $G = \mathrm{SL}_n$, all the maximal parabolic subgroups are minuscule. Similarly, for $G = B_3$ the maximal parabolic P_1 is minuscule and for $G = C_3$, P_3 is minuscule. So, by Lemma 19, the deformed product in the cohomology of the corresponding flag varieties coincides with the usual cup product, so we do not write them here. (The interested reader can find it, e.g., in [KLM].)

Example 1. $G = B_3, P = P_2$:

$H^*(G/P_2)$	b_1	b'_2	b''_2	b'_3	b''_3
b_1	$b'_2 + 2b''_2$	$2b'_3$	$b'_3 + b''_3$	$2\tau b'_4 + \tau b''_4$	$\tau b'_4 + 2\tau b''_4$
b'_2		$2\tau b'_4$	$\tau b'_4 + \tau b''_4$	$2\tau b'_5 + \tau b''_5$	$\tau b''_5$
b''_2			$\tau b'_4 + \tau b''_4$	$\tau b'_5 + \tau b''_5$	$\tau b'_5 + \tau b''_5$
b'_3				$2\tau b_6$	τb_6
b''_3					$2\tau b_6$

$H^*(G/P_2)$	b'_4	b''_4	b'_5	b''_5	b_6	b_7
b_1	$2b'_5 + b''_5$	b''_5	b_6	$2b_6$	b_7	0
b'_2	$2b_6$	0	b_7	0	0	0
b''_2	b_6	b_6	0	b_7	0	0
b'_3	b_7	0	0	0	0	0
b''_3	0	b_7	0	0	0	0

Example 2. $G = B_3, P = P_3$:

$H^*(G/P_3)$	b_1	b_2	b'_3	b''_3	b_4	b_5	b_6
b_1	τb_2	$\tau b'_3 + b''_3$	b_4	τb_4	τb_5	b_6	0
b_2		$2b_4$	b_5	τb_5	b_6	0	0
b'_3			0	b_6	0	0	0
b''_3				0	0	0	0

Example 3. $G = C_3, P = P_1$:

$H^*(G/P_1)$	a_1	a_2	a_3	a_4	a_5
a_1	a_2	τa_3	a_4	a_5	0
a_2		τa_4	a_5	0	0

Example 4. $G = C_3, P = P_2$:

$H^*(G/P_2)$	a_1	a'_2	a''_2	a'_3	a''_3
a_1	$a'_2 + \tau a''_2$	$\tau a'_3$	$a'_3 + a''_3$	$2\tau a'_4 + \tau a''_4$	$\tau a'_4 + 2\tau a''_4$
a'_2		$\tau^2 a'_4$	$\tau a'_4 + \tau a''_4$	$\tau^2 a'_5 + \tau a''_5$	$\tau a''_5$
a''_2			$2a'_4 + 2a''_4$	$\tau a'_5 + 2a''_5$	$\tau a'_5 + 2a''_5$
a'_3				$2\tau a_6$	τa_6
a''_3					$2\tau a_6$

$H^*(G/P_2)$	a'_4	a''_4	a'_5	a''_5	a_6	a_7
a_1	$\tau a'_5 + a''_5$	a''_5	a_6	τa_6	a_7	0
a'_2	τa_6	0	a_7	0	0	0
a''_2	a_6	a_6	0	a_7	0	0
a'_3	a_7	0	0	0	0	0
a''_3	0	a_7	0	0	0	0

Remark 45. We believe that our results can be generalized to the small quantum cohomology of homogenous spaces and the multiplicative eigenvalue problem. In particular, there should be an analogous deformation $\odot_0^q$ of the quantum cohomology of homogenous spaces G/P with an analogous relation to the multiplicative eigenvalue problem ([AW], [Bel1], [Bel4], [TW]).

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